Second Midterm.

General Instructions: Read the statement of each problem carefully. Do only what is requested—nothing more and nothing less. Of course you need not show any work for the multiple choice questions. For the questions that require a written answer, provide a complete solution. If you only write the answer then you will not get full credit.

Be sure to ask questions if anything is unclear. This exam is worth 100 points.

(10 points) 1. The general solution to the differential equation

\[ y'' - 5y' + 4y = 0 \]

is

(a) \( y = Ae^x + Be^{-x} \)
(b) \( y = Ae^{4x} + Be^{-4x} \)
(c) \( y = Ae^x + Be^{4x} \)
(d) \( y = Ae^x + Be^x \)
(e) \( y = Ae^{4x} + Be^{4x} \)

Associated polynomial is

\[ r^2 - 5r + 4 = 0 \]

\[ (r-4)(r-1) = 0 \]

\[ r = 1, 4 \]

Solutions are \( y = e^x, y = e^{4x} \).

General solution is

\[ y = Ae^x + Be^{4x} \]
(10 points) 2. The general solution of the differential equation

\[ y'' - 2y' + 2y = 0 \]

is

(a) \( y = Ae^{-x} \cos x + Be^{-x} \sin x \)

(b) \( y = Ae^{x} \cos x + Be^{x} \sin x \)

(c) \( y = Ae^{x} \cos 2x + Be^{x} \sin 2x \)

(d) \( y = Ae^{-x} \cos(-x) + Be^{-x} \sin(-x) \)

(e) \( y = Ae^{2x} \cos x + Be^{2x} \sin x \)

Associated polynomial is

\[ r^2 - 2r + 2 = 0 \]

\[ r = \frac{2 \pm \sqrt{4 - 4 \cdot 2 \cdot 1}}{2} \]

\[ = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} \]

\[ = 1 \pm i \]

So \( e^{x} \cos x, e^{x} \sin x \) are solutions.

The general solution is

\[ y = Ae^{x} \cos x + Be^{x} \sin x \]
3. The solution of the initial value problem

\[ y'' - 4y' + 4y = 0 \]
\[ y(0) = 1, \ y'(0) = 4 \]

is

(a) \( y = 2e^{2x} + xe^{2x} \)
(b) \( y = e^{2x} + xe^{2x} \)
(c) \( y = xe^{2x} + x^2 e^{2x} \)
(d) \( y = e^{2x} - 2xe^{2x} \)
(e) \( y = e^{2x} + 2xe^{2x} \)

Associated polynomial is

\[ r^2 + 4r + 4 = 0 \]
\[ (r-2)(r+2) = 0 \]
\[ r = 2, -2 \]

\[ So \quad y = e^{2x}, \quad y = xe^{2x} \] are sols.

The general solution is

\[ y = Ae^{2x} + Be^{2x}. \]

For the initial condition,

\[ 1 = y(0) = Ae^{0} + Be^{0} \]
\[ A = 1 \]

\[ y'(x) = 2Ae^{2x} + Be^{2x} + 2Be^{2x} \]
\[ y'(0) = 2A + B + 2B = 0.1 \]
\[ 4 = 2.1 + B \]
\[ B = 2 \]

The solution of the initial value problem is

\[ y = e^{2x} + 2xe^{2x}. \]
(10 points) 4. Using the method of power series, the solution of the differential equation

\[ y' + y = x \]

is

(a) \[ y = a_0 \cdot x + \sum_{j=2}^{\infty} \frac{(-1)^j(1-a_0)}{j!} \cdot x^j \]

(b) \[ y = \sum_{j=2}^{\infty} \frac{(-1)^j}{j!} \cdot x^j \]

(c) \[ y = a_0 \cdot x + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \cdot x^j \]

(d) \[ y = x - 1 + \sum_{j=0}^{\infty} \frac{(-1)^j(1+a_0)}{j!} \cdot x^j \]

(e) \[ y = 2x + \sum_{j=2}^{\infty} \frac{(1+a_0)}{j!} \cdot x^j \]

\[ y = \sum_{j=0}^{\infty} \sum_{l=1}^{j} a_{j-l} \cdot x^j \cdot x^l = \sum_{j=0}^{\infty} \sum_{l=1}^{j} a_{j-l} \cdot x^{j+l} \]

Plugging into the DE gives

\[ \sum_{j=1}^{\infty} j \cdot a_j \cdot x^j + \sum_{j=0}^{\infty} a_j \cdot x^j = x \]

\[ \sum_{j=0}^{\infty} (j+1) \cdot a_{j+1} \cdot x^j + \sum_{j=0}^{\infty} a_j \cdot x^j = x \]

\[ \sum_{j=0}^{\infty} [(j+1) \cdot a_{j+1} + a_j] \cdot x^j = x \]

\[ (a_1 + a_0) \cdot x^0 = 0 \Rightarrow a_1 = -a_0 \]

\[ 2a_2 + a_1 = 1 \Rightarrow a_2 = \frac{1-a_1}{2} = \frac{1+a_0}{2} \]

For \( j \geq 2 \), \( (j+1) \cdot a_{j+1} + a_j = 0 \Rightarrow a_{j+1} = -\frac{a_j}{j+1} \).

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So \[ \frac{dy}{d\ell} = -\frac{\alpha}{3} = \frac{1 + \alpha_0}{3} \]
\[ dy = \frac{\alpha}{3} \ell = \frac{1 + \alpha_0}{3} \ell \]
\[ d\ell = (-1)^j \frac{d^j (1 + \alpha)}{j!} \]

Thus the solution is \[ y = \alpha_0 - \frac{\alpha_0}{2} \ell^2 + \frac{1 + \alpha_0}{2} \ell^2 + \sum_{j=3}^{\infty} \frac{(-1)^j (1 + \alpha)}{j!} \ell^j \]
\[ = \ell - \frac{1}{2} + \sum_{j=0}^{\infty} \frac{(-1)^j (1 + \alpha)}{j!} \ell^j \]

(10 points) 5. One obvious solution of the differential equation

\[ xy'' + 3y' = 0 \]

is the function \( y_1(x) \equiv 1 \). Use the method in the text to find a second solution. What then is the general solution?

(a) \( y = A/x + B/x^2 \)
(b) \( y = Ax + Bx^2 \)
(c) \( y = A + Bx^{1/2} \)
(d) \( y = A + Bx^{-2} \)
(e) \( y = Ax + Bx^{-2} \)

\[ y_2 = \left( \int \frac{1}{y_1} e^{-\int p \ dx} \ dx \right) y_1 \]

Now the DE is \( y'' + \frac{3}{x} y' = 0 \)

So \( p = \frac{3}{x} \int p \ dx = -3 \ln x \)

Thus \( y_2 = \left( \int \frac{1}{x} e^{-3 \ln x} \ dx \right) \cdot 1 \)
\[ = \int \frac{1}{x} \ dx = \frac{1}{x} \]

The general solution is

\[ y = A + B \cdot (-\frac{1}{2}) x^{-2} \]
\[ = A + Bx^{-2} \]
(10 points) 6. Using the method of undetermined coefficients, the general solution of the differential equation

\[ y'' - 3y' + 2y = e^{3x} \]

is

(a) \( y = e^{3x}/2 + Ae^x + Be^{2x} \)
(b) \( y = e^{3x}/4 + Ae^x + Be^{2x} \)
(c) \( y = e^x/8 + Ae^x + Be^{2x} \)
(d) \( y = e^{2x}/8 + Ae^{3x} + Be^x \)
(e) \( y = e^x/8 + Ae^{3x} + Be^{2x} \)

Guess \( y = Ae^{3x} \)

\[ 9 Ae^{3x} - 9 Ae^{3x} + 2 Ae^{3x} = e^{3x} \]
\[ 2 Ae^{3x} = e^{3x} \]
\[ A = \frac{1}{2} \]

So \( y = \frac{1}{2} e^{3x} \) is a particular solution.

The general solution is \( y = \frac{1}{2} e^{3x} + Ae^x + Be^{2x} \)

since \( e^x, e^{2x} \) are solutions of the homogeneous.
Figure 1: A massive cart attached to a wall with a spring.

\[ x'' + \frac{3}{128} x' + \frac{64}{128} x = 0 \]
\[ x'' + \frac{3}{128} x + \frac{1}{2} x = 0 \]

The initial position is \( \frac{1}{2} \) foot so \( x(0) = 0.5 \).
The initial velocity is 0 so \( x'(0) = 0 \).

(10 points) 7. The cart in the figure weighs 128 pounds and is attached to the wall by a spring with spring constant \( k = 64 \text{ lb./ft.} \). The cart is pulled 6 inches in the direction away from the wall and released with no initial velocity. There is a damping force of size \( -3dx/dt \).
The initial value problem that describes this motion is

(a) \( x'' + (64/128)x' + (3/128)x = 0 \), \( x(0) = 6 \), \( x'(0) = 0 \)
(b) \( x'' + x' + x = 0 \), \( x(0) = 1 \), \( x'(0) = 2 \)
(c) \( x'' + (3/64)x' + (3/128)x = 0 \), \( x(0) = -0.5 \), \( x'(0) = 0.5 \)
(d) \( x'' + (3/128)x' + (1/2)x = 0 \), \( x(0) = 0.5 \), \( x'(0) = 0 \)
(e) \( x'' + (3/128)x' + (1/2)x = 0 \), \( x(0) = 6 \), \( x'(0) = 0 \)
(10 points) 8. The radius of convergence of the power series

\[ \sum_{j=0}^{\infty} \frac{x^j}{j^4} \]

is

(a) 1/4
(b) 3
(c) 1/2
(d) 2
(e) 1

\[ \lim_{j \to \infty} \left| \frac{x^{j+1}}{(j+1)^4} \cdot \frac{x^{j+1}}{j^4} \right| = |x| \]

Series converges when |x| < 1.

Radius of convergence is 1.
9. Use the method of variation of parameters to find the general solution of the differential equation

\[ y'' + 4y' + 3y = e^x. \]

\[ y'' + 4y' + 3y = 0 \]
\[ r^2 + 4r + 3 = 0 \]
\[ (r + 1)(r + 3) = 0 \]
\[ r = -1, -3 \]

\[ y_1 = e^{-x}, \quad y_2 = e^{-3x} \]

are solutions to the homogeneous equation.

Seek a solution \( y_0 = v_1 y_1 + v_2 y_2 \) to the inhomogeneous equation.

Solve

\[ y_1 y_1' + y_2 y_2' = 0 \]
\[ y_1' y_1 + y_2' y_2 = e^x \]
\[ e^{-x} v_1' + e^{-3x} v_2' = 0 \]
\[ -e^{-x} v_1' - 3e^{-3x} v_2' = e^x \]
\[ -2e^{-3x} v_2' = e^x \]
\[ v_2' = -\frac{e^x}{2} \]
\[ v_2 = -\frac{e^{2x}}{4} \]

The particular solution is then

\[ y_0 = \frac{e^{2x}}{4} \cdot e^{-x} + (-\frac{e^{2x}}{8} \cdot e^{-3x}) \cdot e^{-x} = \frac{e^x}{4} - \frac{e^x}{8} = \frac{e^x}{8}. \]

The general solution is

\[ y = \frac{e^x}{8} + Ae^{-x} + Be^{-3x} \]
10. Use the method of power series to find the general solution of the differential equation

\[y'' + xy' + y = 0.\]

The ODE becomes

\[\sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} + x \sum_{j=2}^{\infty} j a_j x^{j-1} + \sum a_j x^j = 0\]

\[\sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} + \sum_{j=2}^{\infty} j a_j x^{j-1} + \sum a_j x^j = 0\]

\[\sum_{j=2}^{\infty} j(j+1)a_{j+2} x^j + \sum_{j=2}^{\infty} j a_j x^j + \sum_{j=0}^{\infty} a_j x^j = 0\]

\[\sum_{j=2}^{\infty} [(j+2)(j+1)a_{j+2} + j a_j + a_j] x^j = -2a_2 - a_0\]

\[-2a_2 - a_0 = 0\]

\[a_{j+2}(j+2)(j+1) + (j+1) a_j = 0\]

\[2a_{j+2} = -\frac{a_j}{j+2} \quad j \geq 1\]

So \[a_2 = -\frac{a_0}{2}, \quad a_3 = -\frac{a_1}{3.1}, \quad a_4 = \frac{a_0}{4.2} = \frac{a_0}{2^2.2!}\]

\[a_5 = -\frac{a_2}{5} = \frac{a_1}{5.3.1}, \quad a_6 = -\frac{a_3}{6} = -\frac{a_0}{6.4.2} = -\frac{a_0}{2^3.3!}\]

In general, \[a_{2j} = (-1)^j a_0 \cdot \frac{1}{2j \cdot j!}, \quad a_{2j+1} = (-1)^j a_1 \cdot \frac{1}{1.3.5 \ldots (2j+1)}\]

So the solution is

\[y = \sum_{j=0}^{\infty} \frac{(-1)^j a_0}{2^j \cdot j!} x^j + \sum_{j=0}^{\infty} \frac{(-1)^j a_1}{1.3.5 \ldots (2j+1)} x^{2j+1}\]

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