SECOND MIDTERM EXAM

General Instructions:  Read the statement of each problem carefully. On the multiple choice questions you need not show your work. On the two short answer questions you definitely do need to show your work if you want to receive full credit.

Be sure to ask questions if anything is unclear. This exam is worth 100 points. You will have two hours to take this exam.
(8 points) 1. The general solution to the differential equation

\[ y'' - 5y' + 6y = 0 \]

is

(a) \( Ae^{2x} + Be^{3x} \)
(b) \( Ae^{3x} + Be^{2x} \)
(c) \( Ae^{2x} + Be^{3x} \)
(d) \( e^{3x}(A \sin(2x) + B \cos(2x)) \)
(e) \( e^{2x}(A \sin(3x) + B \cos(3x)) \)

associated polynomial \( t^2 - 5t + 6 = (t-3)(t-2) \)

roots 2, 3

linearly independent solutions \( e^{2x}, e^{3x} \)

general solution \( Ae^{2x} + Be^{3x} \)
(8 points) 2. The general solution of the differential equation

\[ y'' - 2y' + y = 0 \]

is

(a) \( Ae^x + Be^{-x} \)
(b) \( Ae^x + Bxe^{-x} \)
(c) \( Ae^{-x} + Bxe^x \)
(d) \( Ae^x + Bxe^x \)
(e) \( e^x(A \sin x + B \cos x) \)

associated polynomial \( \ell^2 - 2\ell + 1 = (\ell - 1)^2 \)

repeated real root 1

\( \rightarrow \) linearly independent solutions \( e^x, xe^x \)

general solution \( Ae^x + Bxe^x \)
(8 points) 3. Choose the correct solution of the initial value problem:

\[ y'' + 10y' + 26y = 0 \]
\[ y(0) = 1 \]
\[ y'(0) = 0. \]

(a) \( \frac{1}{2}e^{5x} + \frac{1}{2}e^{-x} \)
(b) \( e^{5x} + xe^{-x} \)
(c) \( \cos x + \sin(-5x) \)
(d) \( e^{-5x}(5 \cos x + 5 \sin x) \)
(e) \( e^{-5x}(\cos x + 5 \sin x) \)

The associated polynomial is \( t^2 + 10t + 26 \), roots:

\[ \frac{-10 \pm \sqrt{100 - 4(1)(26)}}{2} = -5 \pm \frac{\sqrt{-41}}{2} = -5 \pm i \]

So the general solution is

\[ y = e^{-5x}(A \cos x + B \sin x) \]

\[ y' = -5e^{-5x}(A \cos x + B \sin x) + e^{-5x}(-A \sin x + B \cos x) \]

\[ y(0) = 1 \implies 1 = A \]

\[ y'(0) = 0 \]

\[ 0 = -5(A + B) + 1(B) \implies B = 5 \]

\[ y = e^{-5x}(\cos x + 5 \sin x) \]
(8 points) 4. We would like to use the method of undetermined coefficients to solve the differential equation

\[ y'' + y = x \cos x. \]

What is an appropriate form of particular solution to try?

(a) \( A \cos x + B \sin x \)
(b) \( A x \cos x + B x \sin x \)
(c) \( A \cos x + B x \cos x + C \sin x + D x \sin x \)
(d) \( A x^2 \cos x + B x^2 \sin x \)
(e) \( A x \cos x + B x^2 \cos x + C x \sin x + D x^2 \sin x \)

we want to try \( y_p = x^s ((A + Bx) \cos x + (C + Dx) \sin x) \)
where \( s \) is the smallest \# such that no terms in the sum are solutions to \( y'' y = 0 \).

Thus we should take \( s = 1 \), which is given (e).
(8 points) 5. One solution to the differential equation

\[ x^2y'' + xy' - y = 0 \]

is \( y_1 = x \). Use the methods from our class to find a second solution.

What is the general solution?

(a) \( Ax + Bx^2 \)
(b) \( Ax + B \)
(c) \( Ax + \frac{B}{x} \)
(d) \( Ax + \frac{B}{x^2} \)
(e) \( Ax + B \cos x \).

Write equation as \( y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0 \).

Then \( v(x) = \int \frac{1}{y_1^2} e^{-\int p(x) \, dx} \, dx \)

\[ = \int \frac{1}{x^2} e^{-\int \frac{1}{x} \, dx} \, dx \]

\[ = \int \frac{1}{x^2} e^{-\ln |x|} \, dx \]

\[ = \int \frac{1}{x^2} \frac{1}{|x|} \, dx = \pm \int \frac{1}{x^3} \, dx \]

so take \( v = -\frac{1}{2} x^{-2} \), and \( vx = -\frac{1}{2} x^{-1} \).

so \( \frac{1}{x} \) is a solution,

and \( Ax + \frac{B}{x} \) is general solution.
(8 points) 6. Use the method of undetermined coefficients to solve

\[ y'' + 4y = 3x^3 \]

(a) \( A \cos(2x) + B \sin(2x) \)
(b) \( A \cos(2x) + B \sin(2x) + 3x^3 \)
(c) \( A \cos(2x) + B \sin(2x) + \frac{3}{4}x^3 - \frac{9}{8}x \)
(d) \( A \cos(2x) + x^3 - \frac{3}{8}x^2 + 3x \)
(e) \( A \cos(2x) + B \sin(2x) + 5x^3 - x^2 + 3x \)

to finish, note that putting together this \( y_p \) and \( A \cos(2x) + B \sin(2x) \) gives the general solution (c) of the nonhomogeneous equation.

The general solution to the homogeneous eqn is \( A \cos 2x + B \sin 2x \).

Try particular solution: \( y_p = Ax^3 + Bx^2 + Cx + D \)
\[ y_p' = 3Ax^2 + 2Bx + C, \quad y_p'' = 6Ax + 2B \]

diff. eq. becomes:

\[ 6Ax + 2B + 4(Ax^3 + Bx^2 + Cx + D) = 3x^3 \]
\[ 4Ax^3 + 4Bx^2 + (6A + 4C)x + (2B + 4D) = 3x^3 + 0x^2 + 0x + 0 \]

\[ 4A = 3, \quad 4B = 0, \quad 6A + 4C = 0, \quad 2B + 4D = 0 \]

so \( A = \frac{3}{4}, \quad B = 0, \quad D = 0, \quad 4C = -6A = -6(\frac{3}{4}) = -\frac{9}{2} \)

\[ \rightarrow C = -\frac{9}{8} \]

\[ y_p = \frac{3}{4}x^3 - \frac{9}{8}x \]

so \( y_p = \frac{3}{4}x^3 - \frac{9}{8}x \).
The cart in Figure 1 weighs 1 kilogram and is attached to the wall by a spring with spring constant $k = 1 \text{ kg/s}^2$. Assume it is not subject to any damping force. If the cart is pulled 2 meters away from the wall and then released, what will be the period of the resulting oscillation (in seconds)?

(a) $\pi$
(b) $2\pi$
(c) $3\pi$
(d) $4\pi$
(e) $5\pi$

The differential equation is $mx'' = -kx$, so $x'' + x = 0$.

$\Rightarrow x(t) = A\cos t + B\sin t$, $x'(t) = -A\sin t + B\cos t$.

$x(0) = 2$, $x'(0) = 0$.

$\Rightarrow A = 2$, $B = 0$.

...so $x(t) = 2\cos t$, and the period of this is $2\pi$. 

(8 points) Now suppose the cart from the previous problem is subject to a damping force \( F_d = -c \frac{dx}{dt} \), where \( c > 0 \). What is the smallest value of \( c \) such that the cart will not oscillate when it is pulled away from the wall and released?

(a) 0  
(b) 1  
\( \textbf{(c)} 2 \)  
(d) 3  
(e) 4  

\[ x'' = -cx' - kx \quad \Rightarrow x'' + cx' + kx = 0 \]

assoc. polynomial \( t^2 + ct + 1, \)

roots \( -c \pm \sqrt{c^2 - 4} \)

we want smallest \( c \) such that \( c^2 - 4 \geq 0 \), which is 2.
(8 points) 9. What is the radius of convergence of the following power series? (Hint: use the ratio test)

\[ \sum_{j=0}^{\infty} \frac{(3x)^j}{j^5} \]

(a) \( \frac{1}{5} \)

(b) \( \frac{1}{3} \)

(c) 1

(d) 3

(e) 5

We have

\[ \lim_{j \to \infty} \left| \frac{(\frac{3x}{j+1})^j}{\frac{(3x)^j}{j^5}} \right| = \lim_{j \to \infty} \left| 3x \left( \frac{j}{j+1} \right)^5 \right| = 13x \]

\( |3x| < 1 \) \iff \( |x| < \frac{1}{3} \) \implies \text{radius of convergence} = \frac{1}{3}
(8 points) 10. Suppose $y_1$ and $y_2$ are two solutions of the differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Which of the following is always a solution of the associated homogeneous equation $y'' + p(x)y' + q(x)y = 0$?

(a) $y_1 + y_2$
(b) $y_1 - y_2$
(c) $y_1 \cdot y_2$
(d) $y_1/y_2$
(e) $y_1^2$

\[
(y_1 - y_2)'' + p(y_1 - y_2)' + q(y_1 - y_2)
= y_1'' + py_1' + qy_1 - (y_2'' + py_2' + qy_2)
= f - f = 0.
\]
11. The general solution of \( y'' + y = 0 \) is \( A \cos x + B \sin x \). Use the method of variation of parameters with \( y_1 = \cos x \) and \( y_2 = \sin x \) to find a particular solution of the equation \( y'' + y = \tan x \).

Want to find \( v_1, v_2 \) such that

\[
\begin{align*}
v_1'y_1 + v_2'y_2 &= 0, \\
v_1'y_1 + v_2'y_2' &= 0
\end{align*}
\]

\[
\begin{align*}
v_1' \cos x + v_2' \sin x &= 0 \\
v_1'(-\sin x) + v_2' \cos x &= \tan x
\end{align*}
\]

\[
\begin{align*}
&\Rightarrow (v_1' \cos^2 x + v_2' \sin x \cos x = 0) \\
&+ (v_1' \sin^2 x + v_2' (-\cos x \sin x) = -\tan x \sin x)
\end{align*}
\]

\[
v_1' = -\tan x \sin x = -\frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x
\]

\[
\begin{align*}
&\Rightarrow (\cos x - \sec x) \cos x + v_2' \sin x = 0 \\
&\cos^2 x - 1 + v_2' \sin x = 0 \quad \Rightarrow -\sin^2 x + v_2' \sin x = 0 \quad \Rightarrow v_2' = \sin x
\end{align*}
\]

So take \( v_1 = \int (\cos x - \sec x) dx = \sin x - \ln |\tan x + \sec x| \)

\[
\begin{align*}
v_2 &= -\cos x, \quad \text{and} \\
y_p &= (\sin x - \ln |\tan x + \sec x|) \cos x + (-\cos x) \sin x = -\cos x \ln |\tan x + \sec x|
\end{align*}
\]
(10 points) 12. Use the method of power series to find the general solution of the differential equation

\[(x^2 - 1)y'' + 4xy' + 2y = 0.\]

\[y = \sum_{n=0}^{\infty} c_n x^n \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} c_n x^{n-2}\]

Diff eq. becomes

\[\sum_{n=2}^{\infty} \frac{n(n-1)}{2} c_n x^{n-2} - \sum_{n=2}^{\infty} \frac{n(n-1)}{2} c_n x^{n-2} + \sum_{n=0}^{\infty} 4n c_n x^n + \sum_{n=0}^{\infty} 2 c_n x^n = 0\]

\[\sum_{n=0}^{\infty} (n+1)(n+2)c_n x^n - \sum_{n=2}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=0}^{\infty} 4n c_n x^n + \sum_{n=0}^{\infty} 2 c_n x^n = 0\]

\[\sum_{n=0}^{\infty} ((n(n-1)+4n+2)c_n - (n+2)(n+1)c_{n+2}) x^n = 0\]

\[\sum_{n=0}^{\infty} ((n^2+3n+2)c_n - (n^2+3n+2)c_{n+2}) x^n = 0\]

\[\rightarrow c_n = c_{n+2}\]

\[\rightarrow y = c_0 \sum_{n=0}^{\infty} x^n + c_1 \sum_{n=0}^{\infty} x^{2n+1} = \frac{c_0 + c_1 x}{1 - x^2}.\]