SOLUTIONS TO MT 3

1. $y'' + \frac{2x-1}{x} y' - \frac{1}{x^2} y = 0$

$x \cdot p = x \cdot \frac{2x-1}{x} = 2x-1$ which is regular $\quad p_0 = -1$

$x^2 \cdot q = x^2 \cdot \left(-\frac{1}{x^2}\right) = -1$ which is regular. $\quad q_0 = -1$

So singular pt. 0 is regular.

$$f(m) = m(m-1) + m p_0 + q_0$$

$$= m(m-1) - m - 1 = 0$$

$$m^2 - m - m - 1 = 0$$

$$m^2 - 2m - 1 = 0$$

$$m = \frac{2 \pm \sqrt{4+4}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

So $m = 1 + \sqrt{2}$, $m = 1 - \sqrt{2}$ are the correct values of m.

2. $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} 2x \, dx = \frac{1}{\pi} x^2 \Big]_{-\pi}^{\pi} = \frac{1}{\pi}\pi^2 - \frac{1}{\pi}\pi^2 = 0.$

For $j \geq 1$, $a_j = \frac{1}{\pi} \int_{-\pi}^{\pi} 2x \cos jx \, dx = \frac{1}{\pi} \frac{\sin jx}{j} \cdot 2x \Big]_{-\pi}^{\pi} - \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin jx}{j} \cdot 2 \, dx$

$$= (0-0) + \frac{1}{\pi} \frac{\cos jx}{j^2} \cdot 2 \Big]_{-\pi}^{\pi}$$

$$= (0-0) + \frac{1}{\pi} \frac{(-1)^j}{j^2} \cdot 2 - \frac{1}{\pi} \frac{(-1)^j}{j^2} \cdot 2$$

$$= 0$$

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} 2x \sin jx \, dx = \frac{2}{\pi}\left(-\frac{\cos jx}{j}\right)\cdot x \Big]_{-\pi}^{\pi} + \frac{2}{\pi}\int_{-\pi}^{\pi} \frac{\cos jx}{j}\, dx$$

$$= \frac{2}{\pi}\left(\frac{-(-1)^j}{j}\cdot \pi + \frac{(-1)^j(-\pi)}{j}\right) + \frac{2}{\pi j}\sin jx \Big]_{-\pi}^{\pi}$$

$$= 4\left(\frac{(-1)^{j+1}}{j}\right) + (0-0).$$

The Fourier series is

$$\sum_{j=1}^{\infty} 4\left(\frac{(-1)^{j+1}}{j}\right)\sin jx.$$

3. Let $\tilde{g}(x) = \begin{cases} x & \text{if } 0 \le x \le \pi \\ -x & \text{if } -\pi \le x < 0. \end{cases}$

Then $\tilde{g}$ is even on $[-\pi, \pi]$. So its Fourier series is a cosine series.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{g}(x)\, dx = \frac{2}{\pi}\int_0^{\pi} x\, dx = \frac{2}{\pi}\frac{x^2}{2}\Big]_0^{\pi}$$

$$= \pi,$$

For $j \ge 1$,

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{g}(x)\cos jx\, dx = \frac{2}{\pi}\int_0^{\pi} x \cos jx\, dx$$

$$= \frac{2}{\pi} x \frac{\sin jx}{j}\Big]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} \frac{\sin jx}{j}\, dx$$

$$= (0-0) + \frac{2}{\pi}\frac{\cos jx}{j^2}\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^j}{j^2} - \frac{1}{j^2}\right] = \frac{2}{\pi j^2}\left[(-1)^j - 1\right].$$

So the cosine series is

$$\frac{\pi}{2} + \sum_{j=1}^{\infty} \frac{2}{\pi j^2}\left[(-1)^j - 1\right]\cos jx,$$

4, $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{1}\int_{-1}^{1} 3x\,dx = \frac{3}{2}x^2\Big]_{-1}^{1} = 0.$

For $j \geq 1$, $a_j = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{j\pi x}{L}\,dx$

$$= \frac{1}{1}\int_{-1}^{1} 3x\cos j\pi x\,dx$$

$$= 3x\frac{\sin j\pi x}{j\pi}\Big]_{-1}^{1} - \int_{-1}^{1}\frac{\sin j\pi x}{j\pi}\cdot 3\,dx$$

$$= (0-0) + \frac{\cos j\pi x}{j^2\pi^2}\cdot 3\Big]_{-1}^{1}$$

$$= \left[\frac{(-1)^j}{j^2\pi^2}\cdot 3 - \frac{(-1)^j}{j^2\pi^2}\cdot 3\right] = 0.$$

$$b_j = \frac{1}{1}\int_{-1}^{1} 3x\sin j\pi x\,dx$$

$$= -\frac{3x\cos j\pi x}{j\pi}\Big]_{-1}^{1} + \int_{-1}^{1}\frac{3\cos j\pi x}{j\pi}\,dx$$

$$= \left[\frac{-3\cos j\pi}{j\pi} + \frac{-3\cos j\pi}{j\pi}\right] + \frac{3\sin j\pi x}{j^2\pi^2}\Big]_{-1}^{1}$$

$$= -\frac{6}{j\pi}(-1)^j + (0-0).$$

So the Fourier series is

$$\sum_{j=1}^{\infty}\frac{-6}{j\pi}(-1)^j\sin j\pi x.$$

5. $\dfrac{1}{p^4-p^2} = \dfrac{1}{p^2(p+1)(p-1)} = \dfrac{A}{p^2} + \dfrac{B}{p+1} + \dfrac{C}{p-1}$

$$= \frac{A(p+1)(p-1) + Bp^2(p-1) + Cp^2(p+1)}{p^2(p+1)(p-1)}$$

$$= \frac{Ap^2 - A + Bp^3 - Bp^2 + Cp^3 + Cp^2}{p^2(p+1)(p-1)}$$

$$= (B+C)p^3 + (A-B+C)p^2 + 0p - A.$$

So $\left.\begin{aligned} B+C&=0 \\ A-B+C&=0 \\ -A&=1 \end{aligned}\right\}$ $\quad \begin{aligned} B+C&=0 \\ -B+C&=1 \\ \hline 2C&=1 \\ C=\tfrac{1}{2} &\Rightarrow B=-\tfrac{1}{2}, \end{aligned}$

Hence

$$\frac{1}{p^4-p^2} = \frac{-1}{p^2} + \frac{-1/2}{p+1} + \frac{1/2}{p-1}.$$

$$L^{-1}\left(\frac{1}{p^4-p^2}\right) = L^{-1}\left(\frac{-1}{p^2}\right) - \frac{1}{2}L^{-1}\left(\frac{1}{p+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{p-1}\right)$$

$$= -x - \frac{1}{2}e^{-x} + \frac{1}{2}e^{x}.$$

6. $L(y''-6y'+9y)=0$

$$(p^2L[y] - p\cdot 0 - 3) - 6(pL[y] - 0) + 9L[y] = 0$$

$$(p^2-6p+9)L[y] = 3$$

$$L[y] = \frac{3}{p^2-6p+9} = \frac{3}{(p-3)^2}$$

So $y = 3xe^{3x}$

7. $f * g(x) = \int_0^x f(x-t)g(t)\,dt$

$$= \int_0^x (x-t)e^{-t}\,dt$$

$$= x\int_0^x e^{-t}\,dt - \int_0^x te^{-t}\,dt$$

$$= x(-e^{-t})\Big]_0^x + te^{-t}\Big]_0^x - \int_0^x e^{-t}\,dt$$

$$= -xe^{-x} + x + xe^{-x} - 0 + e^{-t}\Big]_0^x = e^{-x} - 1 + x.$$

8. $z(p) = p^2 + p - 2$

$$L[h] = \frac{1}{z(p)} = \frac{1}{p^2+p-2} = \frac{1}{(p+2)(p-1)} = \frac{A}{p+2} + \frac{B}{p-1}$$

$$= \frac{A(p-1) + B(p+2)}{(p+2)(p-1)}$$

$$= \frac{(A+B)p + (2B - A)}{(p+2)(p-1)}$$

$$A + B = 0$$
$$2B - A = 1$$
$$3B = 1 \Rightarrow B = \tfrac{1}{3},\ A = -\tfrac{1}{3}.$$

So $L[h] = \dfrac{-1/3}{p+2} + \dfrac{1/3}{p-1}$

$$h = -\tfrac{1}{3}e^{-2x} + \tfrac{1}{3}e^{x}$$

So $y(t) = \int_0^t h(t-\tau)f(\tau)\,d\tau$

$$= \int_0^t \Big[-\frac{1}{3}e^{-2(t-\tau)} + \frac{1}{3}e^{t-\tau}\Big]\tau\,d\tau$$

$$= -\frac{1}{3}e^{-2t}\int_0^t e^{2\tau}\cdot\tau\,d\tau + \frac{1}{3}e^{t}\int_0^t e^{-\tau}\cdot\tau\,d\tau$$

$$= \Big(-\frac{1}{3}e^{-2t}\Big\{\Big[\frac{e^{2\tau}}{2}\cdot\tau\Big]_0^t - \int_0^t \frac{e^{2\tau}}{2}\,d\tau\Big\}\Big)$$
$$+ \Big(\frac{1}{3}e^{t}\Big\{-e^{-\tau}\cdot\tau\Big]_0^t + \int_0^t e^{-\tau}\,d\tau\Big\}\Big)$$

$$= \Big(-\frac{1}{3}e^{-2t}\Big\{\Big[\frac{e^{2t}}{2}t - 0\Big] - \frac{e^{2\tau}}{4}\Big]_0^t\Big\}\Big)$$
$$+ \Big(\frac{1}{3}e^{t}\Big\{\Big[-e^{-t}\cdot t + 0\Big] - e^{-\tau}\Big]_0^t\Big\}\Big)$$

$$= \Big(-\frac{1}{3}e^{-2t}\Big\{\frac{e^{2t}t}{2} - \frac{e^{2t}}{4} + \frac{1}{4}\Big\}\Big) + \Big(\frac{1}{3}e^{t}\Big\{-te^{-t} - e^{-t} + 1\Big\}\Big)$$

$$= -\frac{1}{3}\cdot\frac{t}{2} + \frac{1}{12} - \frac{1}{12}e^{-2t} - \frac{1}{3}t - \frac{1}{3} + \frac{1}{3}e^{t}$$

$$= -\frac{1}{6}t - \frac{1}{12}e^{-2t} + \frac{1}{3}e^{t} - \frac{1}{4} - \frac{t}{3}$$

$$= -\frac{1}{2}t - \frac{1}{12}e^{-2t} + \frac{1}{3}e^{t} - \frac{1}{4}.$$

9, $f(-x) = (\sin -x)\cdot(\sin -x)$

$= (-\sin x)\cdot(-\sin x) = (\sin x)\cdot\sin x$

so the function is even.

10. $\frac{1}{\pi}\int_{-\pi}^{\pi} \sin jx \sin kx \, dx =$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(j-k)x - \cos(j+k)x}{2} \, dx$

$= \frac{1}{\pi}\left[\frac{\sin(j-k)x}{2(j-k)} - \frac{\sin(j+k)x}{2(j+k)}\right]_{-\pi}^{\pi}$

$= 0.$

11. $\frac{d}{dp}F(p) = \frac{d}{dp}\int_0^\infty e^{-px} f(x)\,dx$

$= \int_0^\infty \frac{d}{dp} e^{-px} f(x)\,dx$

$= \int_0^\infty e^{-px}\left[-x f(x)\right] dx$

$= L\left[-x f(x)\right].$

12. $L[x\cos x] = -L[-x\cos x]$

$= -\frac{d}{dp} L[\cos x]$

$= -\frac{d}{dp}\frac{p}{p^2+1} = -\left[\frac{(p^2+1)\cdot 1 - p\cdot 2p}{(p^2+1)^2}\right]$

$= -\left[\frac{p^2+1-2p^2}{(p^2+1)^2}\right] = \frac{p^2-1}{(p^2+1)^2}.$