Math 310
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More on Topology

The last example—which we sometimes call the *trivial topology*—will work on any nonempty set X. Equip X with just the two open sets X and $\emptyset$. That is a topology.

Example:

Let X be the set of integers $\mathbb{Z}$. Call a set open if it is either empty or all of $\mathbb{Z}$ or is the complement of a finite set. Then it is straightforward to confirm that this is a topological space. In particular, note that the union of sets with a finite complement will still be a set with a finite complement. And the finite (certainly *not* the infinite) intersection of sets with finite complement will be a set with a finite complement.

We may note that the last example works for any infinite set X (not just the integers $\mathbb{Z}$).

Example:

Consider the topology on the real line generated by intervals of the form $[a, b)$ or $[a, +\infty)$ (here we mean "generated" in the sense of taking finite intersection and arbitrary union). This is called the *Sorgenfrey line*, named after Robert Sorgenfrey (1915–1996). The Sorgenfrey line is one of the most important examples in topology.

First note that, if (c, d) is any open interval, then

$$(c, d) = \bigcup_{\epsilon > 0} [c + \epsilon, d) \,.$$

Thus (c, d) is the union of Sorgenfrey open sets. So any standard open interval is open in the Sorgenfrey topology. We see, then, that the Sorgenfrey topology contains all the usual open sets and some new ones as well.

We shall encounter the Sorgenfrey line at several junctures in the sequel; it is an important example in several contexts of point-set topology.

If $(X, \mathcal{U})$ is a topological space, then $(Y, \mathcal{V})$ is a *topological subspace* (or more simply a *subspace*) if $Y \subset X$ and each $V \in \mathcal{V}$ is of the form $V = Y \cap U$ for some $U \in \mathcal{U}$.

In a topological space $(X, \mathcal{U})$, a set $E \subset X$ is called *closed* if its complement $X \setminus E$ is open. In the first example, any interval $[a, b]$ is closed (though these are certainly not all the closed sets!—see our discussion of the Cantor set). In the example of the trivial topology on $[0, 1]$, the only closed sets are the entire interval $[0, 1]$ and the empty set. In the next example, the closed sets are the finite sets (and the empty set and the entire space X).

There are sets that are neither open nor closed, such as the interval $[0, 1)$. It is also possible for a set to be both open and closed. In any topological space $(X, \mathcal{U})$, the entire space X and the empty set $\emptyset$ are both open and closed.

Proposition:

The union of two closed sets is closed.

Proof: Let the two closed sets be E and F. Then $X \setminus E$ and $X \setminus F$ are open. So

$$S \equiv (X \setminus E) \cap (X \setminus F)$$

is open. But then

$$^{c}S \equiv X \setminus S = E \cup F$$

is closed. $\square$

Proposition:

Let $\{E_\beta\}_{\beta \in B}$ be closed sets. Then $\cap_\beta E_\beta$ is also closed.

Proof: Just take complements and use the fact that any union of open sets is open. $\square$

We want to develop a language for describing and analyzing the parts of a set. Consider the closed disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ depicted in the first figure below. We can see intuitively that this set has a boundary (the circle—see the second figure below). And it has an interior (the open disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$—see the third figure below). We would like a precise description of the boundary and interior of *any* set. For example, consider the set S of integers in the topological space $\mathbb{R}$ with the usual topology, as described in an earlier example. What is its interior and what is its boundary? The answer to this last question is not obvious, and requires some study.

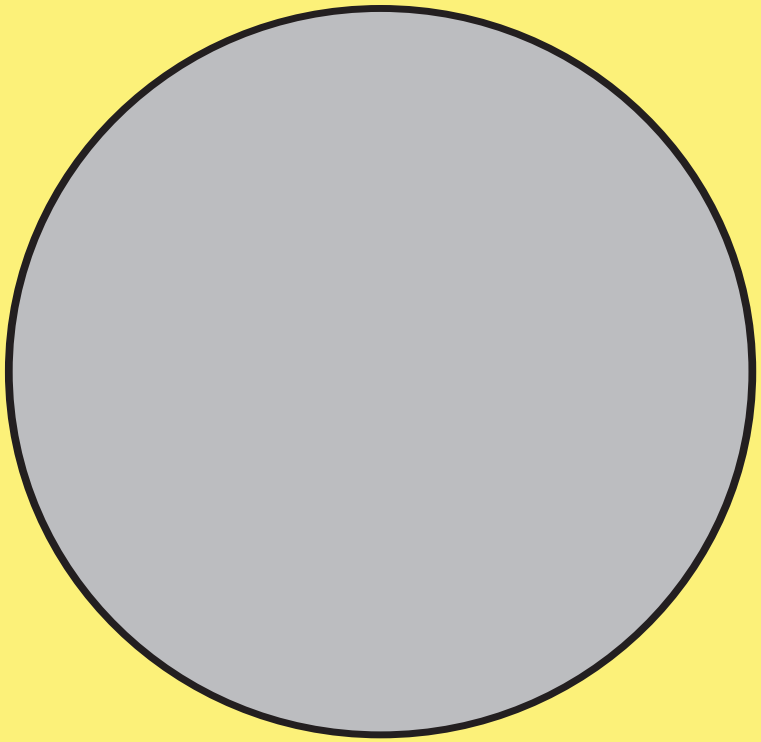


Figure: The closed disc.

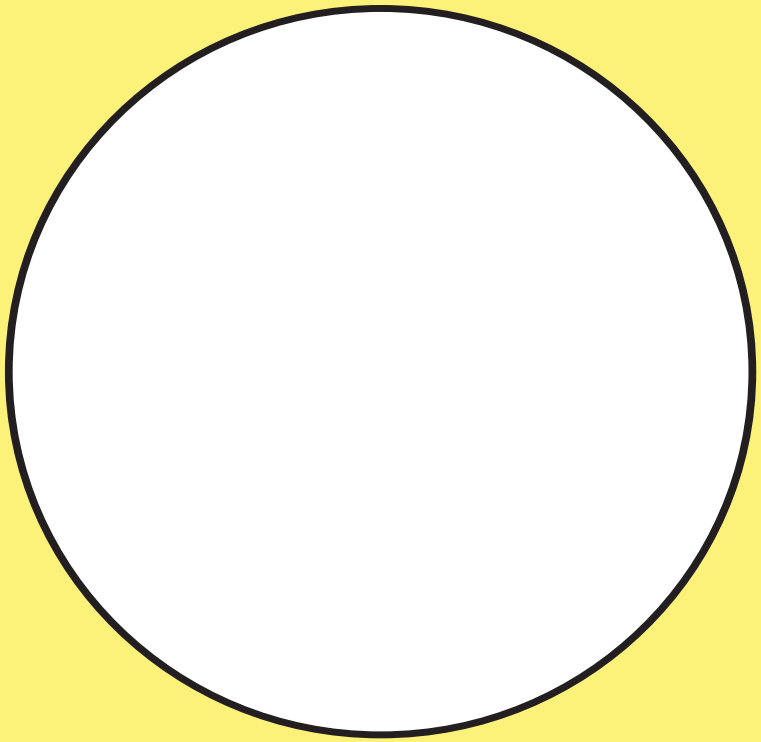


Figure: The boundary of the disc.

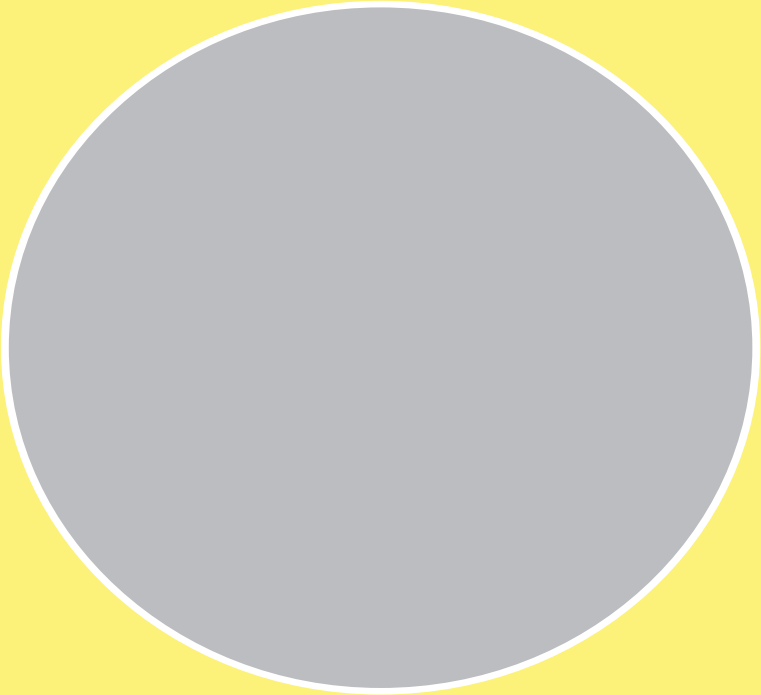


Figure: The interior of the disc.

First we need a bit of terminology. If X is a topological space and $x \in X$, then a *neighborhood* U of x is an open set that has x as an element.

Definition:

Let S be any set in the topological space X. A point $p \in X$ is said to be an *interior point* of S if

(a) $p \in S$,

(b) There is a neighborhood U of p such that $p \in U \subset S$.

Definition:

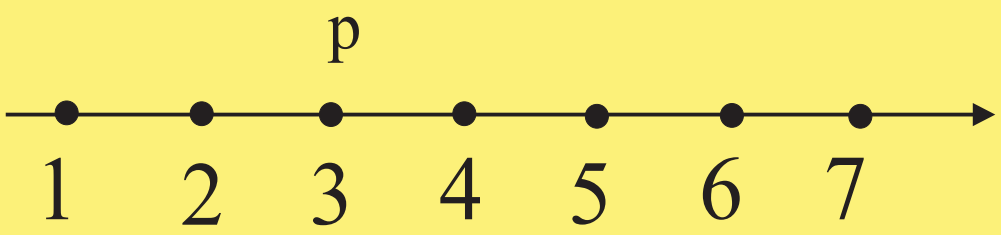
Let S be any set in the topological space X. A point $q \in X$ is said to be a *boundary point* of S if any neighborhood U of q intersects both S and ${}^cS \equiv X \setminus S$.

These new concepts can be elucidated through some examples.

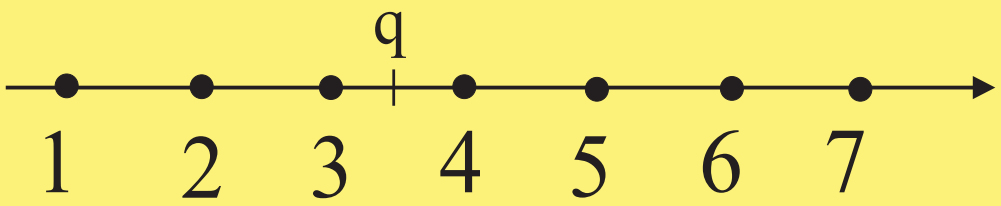
Example:

Let us return to the example of the integers $\mathbb{Z}$ as a subspace of the real line equipped with the Euclidean topology. If p is any point of $\mathbb{Z}$ and U is any neighborhood of p, then it is clear that U will contain points that do not lie in $\mathbb{Z}$. See the first figure. Therefore there is no point that is an interior point of $\mathbb{Z}$, so the interior of $\mathbb{Z}$ is empty.

If q is any point of ${}^c\mathbb{Z}$, then let $\epsilon > 0$ be the distance of q to the nearest integer. See the second figure. Then the interval $V = (q - \epsilon, q + \epsilon)$ is a neighborhood of q that intersects ${}^c\mathbb{Z}$ but does not intersect $\mathbb{Z}$ itself. So q cannot be a boundary point of $\mathbb{Z}$. If instead q is an element of $\mathbb{Z}$ and U is any neighborhood of q, then $q \in U \cap \mathbb{Z}$ and, plainly, U will also contain nearby points that are not in $\mathbb{Z}$ (see the first figure). Thus q is a boundary point of $\mathbb{Z}$. We see then that the boundary of $\mathbb{Z}$ is just $\mathbb{Z}$ itself.



The point p is in the boundary of $\mathbb{Z}$.



The point q is not in the boundary of $\mathbb{Z}$.

If S is a set in a topological space X, then we denote the interior of S by $\overset{\circ}{S}$ and the boundary of S by ∂S.

Example:

Consider X the real line with the usual topology. Let S be the set $[0, 1)$. Then the boundary of S is the pair $\partial S = \{0, 1\}$, and the interior is the interval $\overset{\circ}{S} = (0, 1)$.

Example:

For the interval $[0, 1]$ equipped with the trivial topology, let S be any proper subset of the interval $X = [0, 1]$. If p is any point of S, then the only possible neighborhood of p is the entire interval $[0, 1]$ because that is the only open set available. Since that interval cannot lie in S, we conclude that S has no interior points.
Now if q is *any point* in the entire space X, then the only neighborhood of q is the entire interval $[0, 1]$ (since that is the only open set available). And that interval intersects both S and its complement. We conclude that ∂S is the entire space $[0, 1]$.

The interior and the boundary of a set have many interesting properties. We shall record a few of them here. We begin with a lemma that has independent interest.

Lemma:

Let S be a set in a topological space X. If each point $s \in S$ has a neighborhood that lies in S, then S is open.

Proof: Let $s \in S$ and let U_s be the neighborhood of s that lies in S. We have

$$S = \bigcup_{s \in S} \{s\} \subset \bigcup_{s \in S} U_s \subset S\,.$$

Since the far left and the far right in these last containments is the same, we conclude that

$$\bigcup_{s \in S} U_s = S\,.$$

But the set on the left of this last equality, being the union of open sets, is open. Hence S itself is open. $\square$