Math 310
April 17, 2020 Lecture

Steven G. Krantz

March 30, 2020

Moving Ahead in Topology

In practice in this discussion, and in topology in general, when we say "Let $f : X \to Y$ be a mapping," we mean that f is a *continuous* mapping. We shall follow that custom consistently in what follows.

The Separation Axioms

The richness of the subject of topology begins to become evident when we examine and classify the different types of topological spaces. We do so by way of the *separation axioms*.

We begin with a sample separation axiom that is particularly intuitively appealing—just to give a flavor of this circle of ideas.

Definition:

We say that a nonempty topological space X is a *Hausdorff space* if, for any two distinct points P, Q in X, there are open sets U and V such that

- $P \in U$ and $Q \in V$,
- $U \cap V = \emptyset$.

Example:

Let X be the real line with the usual topology. This is a Hausdorff space. For if P and Q are distinct points in $\mathbb{R}$ and if $\epsilon = |P - Q|$, then the intervals $U = (P - \epsilon/3, P + \epsilon/3)$ and $V = (Q - \epsilon/3, Q + \epsilon/3)$ are neighborhoods of P and Q that are disjoint.

Example:

Let X be the integers with the topology that U is open if it is the complement of a finite set (or else it is the empty set or the whole space). Then this X is *not* a Hausdorff space. For if P, Q are distinct points, and if U, V are neighborhoods of P and Q, respectively, then $U \cap V$ will always be an infinite set, and thus not empty.

The separation axioms are so important that they are numbered. Traditionally we call a Hausdorff space a $\mathbf{T_2}$ space. Let us now lay out all the separation axioms. We begin with structures that are weaker than Hausdorff, and end with structures that are stronger. We shall use the terminology "neighborhood of a set S" to mean an open set that contains S.

$\mathbf{T_0}$ Space: The nonempty space X is $\mathbf{T_0}$ if, whenever $P, Q \in X$ are distinct points, then either there is a neighborhood U of P such that $Q \notin U$ or else there is a neighborhood V of Q such that $P \notin V$.

$\mathbf{T_1}$ Space: The nonempty space X is $\mathbf{T_1}$ if, whenever $P, Q \in X$ are distinct points, then there are neighborhoods U of P and V of Q such that $Q \notin U$ and $P \notin V$. It is easy to check that, in a $\mathbf{T_1}$ space X, any singleton set $\{x\}$ will be closed.

$\mathbf{T_2}$ Space: The nonempty space X is $\mathbf{T_2}$ (also called *Hausdorff*) if, whenever $P, Q \in X$ are distinct points, then there are neighborhoods U of P and V of Q such that $U \cap V = \emptyset$.

T_3 Space: The nonempty space X is $\mathbf{T_3}$ (also called *regular* if singleton sets are closed) if, whenever $P \in X$ and $F \subset X$ is a closed subset not containing P, then there are neighborhoods U of P and V of F so that $U \cap V = \emptyset$.

T_4 Space: The nonempty space X is $\mathbf{T_4}$ (also called *normal* if singleton sets are closed) if, whenever E and F are disjoint closed sets in X, then there are neighborhoods U of E and V of F such that $U \cap V = \emptyset$.

Let us examine some examples that show that these separation axioms define distinct spaces. The entire subject of point-set topology is built on examples like these. You should master them and make them part of your toolkit. It should be noted that the separation axioms (at least for spaces that are assumed to be $\mathbf{T_1}$) increase in strength as the index increases. So a $\mathbf{T_3}$ (regular) space is certainly $\mathbf{T_2}$ (Hausdorff). But not conversely.

Example:

Let X be the real line with the open sets being the half-lines of the form (a, ∞). It is clear that the collection $\mathcal{U}$ of such half-lines is closed under union and finite intersection. If we throw in the whole space and the empty set, then we certainly have a topology.

This space is $\mathbf{T_0}$ but not $\mathbf{T_1}$. To see this, note that if P and Q are distinct points of X and if $P < Q$, then $U = \{x : x > P\}$ is an open set in X. Also $Q \in U$ but $P \notin U$. So certainly X is $\mathbf{T_0}$. But it is easy to see that there is no open neighborhood of P that will separate it from Q. So X is not $\mathbf{T_1}$.

Example:

Let X be the integers equipped with the topology that U is open if its complement is finite (together with the empty set and the whole space). We have already seen that this space is not Hausdorff. It is, however, of type $\mathbf{T_1}$; because if $P, Q \in X$ are distinct, then let U be the complement of $\{Q\}$ and let V be the complement of $\{P\}$. Since the space is $\mathbf{T_1}$, it is also $\mathbf{T_0}$.

Example:

Let X be the real line and equip it with the following topology. If $x \in X$ is a point other than 0, then let the neighborhoods of x be the usual intervals $U_{x,\beta} = (x - \beta, x + \beta)$ for $\beta > \mathbf{0}$. If $x = \mathbf{0}$, then let a neighborhood of x have the form

$$U_{\mathbf{0},\alpha} = \{t \in \mathbb{R} : -\alpha < t < \alpha, t \neq \mathbf{1}, \mathbf{1/2}, \mathbf{1/3}, \dots\}$$

for $\alpha > \mathbf{0}$. Now generate a topology by taking all finite intersections and arbitrary unions of the described sets $U_{x,\beta}$ and $U_{\mathbf{0},\alpha}$. It is easy to see that the resulting space is $\mathbf{T_2}$, because any two distinct points can be separated by intervals in the usual fashion—i.e., if x and y are distinct points, then there are disjoint open sets U and V such that $x \in U$ and $y \in V$.

But the space is not $\mathbf{T_3}$, because $E = \{\mathbf{1}, \mathbf{1/2}, \mathbf{1/3}, \dots\}$ is a closed set in this topology and it cannot be separated from the point 0 with open sets.

Example:

The *Moore plane* $\mathcal{P}$ (named after R. L. Moore (1882–1974)) is the usual closed upper halfplane $\{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}, y \geq \mathbf{0}\}$ with the topology generated by these open sets: **(a)** If $(x, y) \in \mathcal{P}$ and $y > \mathbf{0}$, then consider any disc of the form $\{(s, t) \in \mathcal{P} : (s - x)^{\mathbf{2}} + (t - y)^{\mathbf{2}} < r^{\mathbf{2}}\}$ for $r < y$; **(b)** If $(x, \mathbf{0}) \in \mathcal{P}$, then consider the singleton $\{(x, \mathbf{0})\}$ union any disc of the form $\{(s, t) : (s - x)^{\mathbf{2}} + (t - r)^{\mathbf{2}} < r^{\mathbf{2}}\}$ for $r > \mathbf{0}$. We generate the topology by taking finite intersections and arbitrary unions of the sets described in **(a)** and **(b)**. This space is $\mathbf{T_3}$ but not $\mathbf{T_4}$.

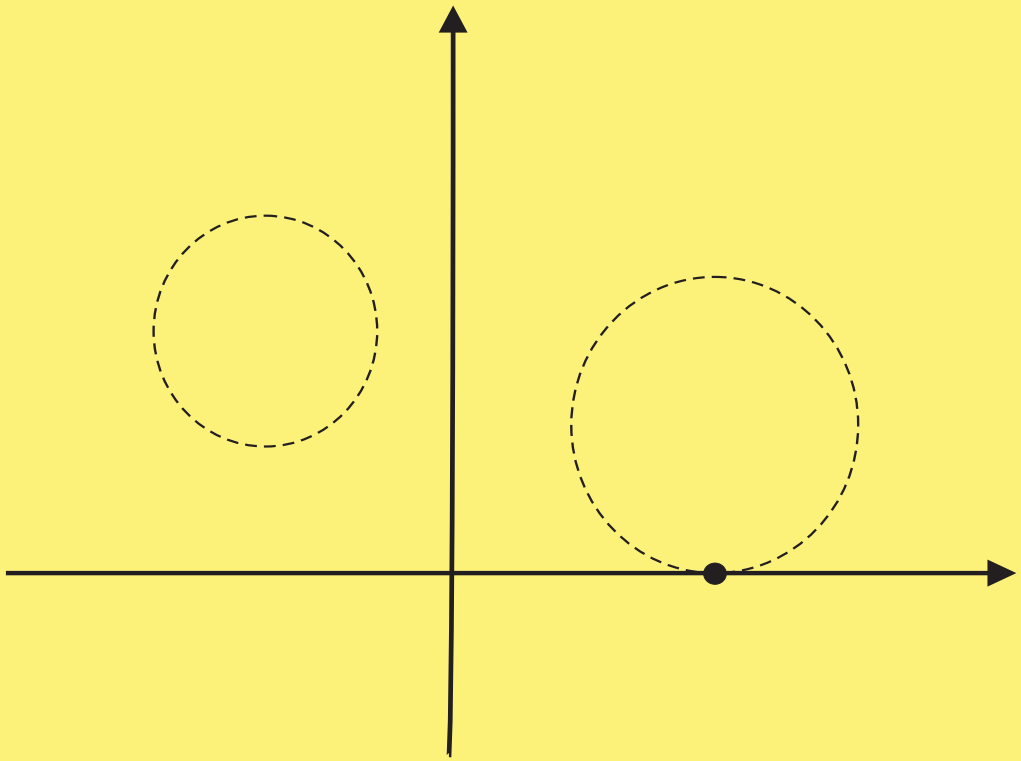

Figure: Open sets in the Moore plane.

To see that the space is $\mathbf{T_3}$, first note that any subset of the real axis is closed. But it can easily be separated by open sets from a disjoint point in the real axis using the special open sets of type **(b)**. Points in the open upper halfplane are separated from closed sets in $\mathcal{P}$ in the usual fashion.

For nonnormality, examine $E = \{(x, \mathbf{0}) \in \mathcal{P} : x \text{ is rational}\}$ and $F = \{(x, \mathbf{0}) \in \mathcal{P} : x \text{ is irrational}\}$. Then each of these sets is closed. They are clearly disjoint, but they cannot be separated by open sets. Prove these statements as an exercise.

Example:

Consider the interval $X = [\mathbf{0}, \mathbf{1}] \subset \mathbb{R}$ with this topology: A set $V \subset [\mathbf{0}, \mathbf{1}]$ is open if there is an open set $U \subset \mathbb{R}$ (in the usual Euclidean topology) such that $U \cap [\mathbf{0}, \mathbf{1}] = V$. This X is a $\mathbf{T_4}$ space. To see this, let E and F be disjoint closed sets in $[\mathbf{0}, \mathbf{1}]$. We claim that there is a number $\delta > \mathbf{0}$ such that if $e \in E$ and $f \in F$, then $|e - f| > \delta$. If not, then there are $e_j \in E$ and $f_j \in F$ with $|e_j - f_j| \to \mathbf{0}$. But then the common limit point x of the two sequences $\{e_j\}$ and $\{f_j\}$ would have to lie in the boundary both of E and of F. Since E and F are disjoint closed sets, that is impossible. So δ exists. Now let $U = \{x \in X : \text{dist}(x, E) < \delta/\mathbf{3}\}$ and $V = \{x \in X : \text{dist}(x, F) < \delta/\mathbf{3}\}$. Here $\text{dist}(x, S)$ denotes the distance of x to the set S, defined to be $\inf_{s \in S} |x - s|$. Of course $E \subset U$ and $F \subset V$. It is easy to see that U, V are both open, and they are disjoint by the triangle inequality.

We refer the reader to our earlier discussions for further consideration of some of the ideas in the last example. Section 8.5 also develops the idea of compactness.

One of the most important applications of the idea of separation is the following basic result of Urysohn (commonly known as *Urysohn's lemma*—Pavel Urysohn (1898–1924)):

Theorem

Let $(X, \mathcal{U})$ be a normal space and let E and F be disjoint, closed sets in X. Then there is a continuous function $f : X \to [\mathbf{0}, \mathbf{1}]$ such that $f(E) = \{\mathbf{0}\}$ and $f(F) = \{\mathbf{1}\}$ (that is, the image of each point in E is $\mathbf{0}$ and the image of each point of F is $\mathbf{1}$).

Remark:

What is interesting, and significant, about Urysohn's lemma is that it relates topology to the theory of functions. In a sense, the function f in the theorem is "separating" the sets E and F.

Proof of the Theorem: By normality, there are disjoint open sets U and V such that $E \subset U$ and $F \subset V$. For technical (and also traditional) reasons, we shall denote this set U by $U_{\mathbf{1/2}}$. Now we see that E and $X \setminus U_{\mathbf{1/2}}$ are closed and disjoint. Also $\overline{U}_{\mathbf{1/2}}$ and F are closed and disjoint. Therefore open sets $U_{1/4}$, $U_{3/4}$ exist such that

$$E \subset U_{1/4} \ , \ \overline{U}_{1/4} \subset U_{1/2} \ , \ \overline{U}_{1/2} \subset U_{3/4} \ , \ \overline{U}_{3/4} \cap F = \emptyset \, .$$

Suppose now inductively that sets $U_{j/2^n}$, $j = 1, 2, \ldots, 2^{n-1}$, have been defined so that

$$E \subset U_{1/2^n} \ , \ \ldots \ , \ \overline{U}_{(j-1)/2^n} \subset U_{j/2^n} \ , \ \ldots \ , \ \overline{U}_{2^n-\mathbf{1})/\mathbf{2}^n} \cap F = \emptyset \, .$$

Then we may continue and select sets $U_{j/2^{n+1}}$, $j = 1, \ldots, 2^{n+1} - 1$ with analogous properties.

The result of our construction is that we have, for each dyadic rational number r of the form $j/2^n$, for some $n > 0$ and $j = 1, 2, \ldots, 2^n - 1$, an open set U_r satisfying

- $E \subset U_r$ and $\overline{U}_r \cap F = \emptyset$,
- $\overline{U}_r \subset U_s$ whenever $r < s$ are dyadic as above.

Now define a function $f : X \to [0,1]$ by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \text{ belongs to no } U_r\,; \\ \inf\{r : x \in U_r\} & \text{if} \quad x \text{ belongs to some } U_r\,. \end{cases}$$

Then $f(E) = 0$ and $f(F) = 1$. It remains to show that f is continuous.

We have

Continuity at points x with $f(x) = 1$: If $x \notin \overline{U}_r$, then $f(x) \geq r$.

Continuity at points x with $f(x) = 0$: If $x \in U_r$, then $f(x) \leq r$.

Continuity at all other points: If $x \in U_s \setminus \overline{U}_r$, where $r < s$ are dyadic, then $r \leq f(x) \leq s$.

The existence of the continuous function f is now established. $\square$