Final Exam

General Instructions: Read each problem carefully. Do only what is requested—nothing more nor less. Always show your work on each problem—unless the problem explicitly tells you not to show work. Always give mathematical reasons for your answers. You will not get full credit for just writing down the answer. It is always best to show full details. The point value of each problem is shown. The sum of all the point values on this exam is 200. Ask questions if anything is unclear. Use the backs of the pages if you need extra space.

1. (10 points) What is the converse of the contrapositive of the statement

If it is cloudy then it is raining.

The contrapositive is

If it is not raining then it is not cloudy.

The converse of that is

If it is not cloudy then it is not raining.
2. (10 points) Prove by induction that
\[ 2n < 1 + n^2 \]
for all natural numbers \( n > 1 \).

For \( n = 2 \) the statement is clear.

Assume it is established for \( n = j \). So
\[ 2j < 1 + j^2, \]

Hence
\[ 2j + 2 < 1 + j^2 + 2 \leq 1 + j^2 + 2j \]
\[ < (1 + j)^2 + 1 \]
\[ \therefore 2(j+1) < 1 + (j+1)^2. \quad \Box \]

3. (10 points) Prove by contradiction that if the natural number \( n \) is a perfect square then \( n + 1 \) cannot be a perfect square.

If \( n = k^2 \) and \( n + 1 = l^2 \) then
\[ 1 = (n+1) - n = l^2 - k^2 = (l-k)(l+k), \]
Thus \( l-k = 1 \)
\[ l + k = 1. \]

So \( l = 1 \), \( k = 0 \) which is a contradiction. \( \Box \)
4. (10 points) Prove that, if \( A \) and \( B \) are sets, then
\[
(A \cap B)' = A' \cup B'.
\]
If \( x \in (A \cap B)' \) then \( x \notin A \cap B \) so either \( x \notin A \) or \( x \notin B \). Hence \( x \in A' \) or \( x \in B' \), so \( x \in A' \cup B' \).

Conversely, if \( x \in A' \cup B' \) then either \( x \in A' \) or \( x \in B' \), so \( x \notin A \cap B \). Hence \( x \in (A \cap B)' \).

5. (10 points) Draw a Venn diagram to illustrate the identity
\[
(A \cup B)' = (A' \cap B').
\]
6. (10 points) Is the collection of all sets of real numbers with two elements a countable set or an uncountable set? Why or why not?

Consider the set $S$. Then we have a one-to-one function

$$\varphi : \mathbb{R} \to S$$

$$\varphi (x) = \{ x, 1 \}$$

So $\text{card}(\mathbb{R}) \leq \text{card}(S)$.

Hence $S$ is uncountable.

7. (10 points) Which of these sets are countable or uncountable or finite or empty? Give brief reasons for your answers.

(a) $\mathbb{N} \times \mathbb{Q}$

$\mathbb{N}$ and $\mathbb{Q}$ are countable so $\mathbb{N} \times \mathbb{Q}$ is countable.

(b) $\mathbb{C} \times \mathbb{R}$

$\mathbb{C}$ and $\mathbb{R}$ are uncountable so $\mathbb{C} \times \mathbb{R}$ is uncountable.

(c) $\mathbb{C} \setminus \mathbb{R}$

$$\varphi : \mathbb{R} \to \mathbb{C} \setminus \mathbb{R}$$

$$x \mapsto \text{fix}$$

is one-to-one. So $\text{card}(\mathbb{R}) \leq \text{card}(\mathbb{C} \setminus \mathbb{R})$.

Hence $\mathbb{C} \setminus \mathbb{R}$ is uncountable.
(d) \( \mathbb{Q} \setminus \mathbb{N} \) is countable. \( \mathbb{Q} \setminus \mathbb{N} \) contains all fractions \( \frac{n}{n+1} \), so \( \mathbb{Q} \setminus \mathbb{N} \) is countable.

(e) \( \cap_{j=1}^{\infty} \{ x \in \mathbb{R} : x > j \} \)

There is no real number \( x \) which is greater than every \( j \). This set is empty.

8. (10 points) Use the Axiom of Regularity to show that no set can be an element of itself.

Let \( A \) be a set. Apply the Axiom of Regularity to \( \{ A \} \). So \( A \) is an element of \( \{ A \} \) which is disjoint from \( \{ A \} \). Since the only element of \( \{ A \} \) is \( A \), we see that \( A \) is disjoint from \( \{ A \} \). So \( A \in A \).
9. (10 points) Find all square roots of \( k \) in the quaternions.

\[(a + bi + cj + dk)(a + bi + cj + dk) = k\]
\[a^2 + abi + acj + adk + ab^2 + bci + bdj + c^2 + cdi + ade + bde = k\]
\[a^2 + b^2 + c^2 + d^2 - 2bc - c^2 + d^2 = 0, \quad ab + 2bc + ad - cd = 0, \quad ac - bd + 2c + bd = 0\]
\[ad + bc - bc + 2d = 1 \quad \text{Cannot have } a = 0 \text{ since then } b = c = d = 0\]

So \( a \neq 0 \Rightarrow d = \frac{1}{2a} \), \( c = 0 \), \( b = 0 \). Hence

\[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}k \quad \text{and} \quad -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}k \quad \text{are the square roots}.

10. (10 points) Prove that multiplication in the rational numbers \( \mathbb{Q} \) is well defined.

Assume that \((a, b) \sim (a', b'), (c, d) \sim (c', d')\).

Need to know that

\[(a, b) \cdot (c, d) \sim (a', b') \cdot (c', d') \quad \text{or} \quad (ac, bd) \sim (a'c', b'd').\]

But this is \(acbd' = a'c'bd\). (*)

We assume that \(ab = a'b'\) and \(cd = c'd\),

Multiply these two together to get (*).
11. (10 points) Explain how we define the natural numbers 1, 2, 3, 4 in terms of set theory.

We set

\[ 1 = \{ \emptyset \} \]
\[ 2 = 1 \cup \{ 1 \} \]
\[ 3 = 2 \cup \{ 2 \} \]
\[ 4 = 3 \cup \{ 3 \} \]

12. (10 points) Let \( G \) be a group with finitely many elements and \( H \) a subgroup. Explain why the order of \( H \) (i.e., the number of elements in \( H \)) evenly divides the order of \( G \) (i.e., the number of elements in \( G \)).

Let \( G \) have \( n \) elements and \( H \) have \( k \) elements. Then each coset of \( H \) has \( k \) elements. And the disjoint union of the cosets in \( G \), hence \( n \), is a multiple of \( k \).
13. (10 points) Let $E_1 \supset E_2 \supset E_3 \supset \cdots$ be countably many nonempty, compact sets. Prove that $\bigcap_{j=1}^{\infty} E_j$ is compact and nonempty.

Each $E_j$ is closed and bounded so the intersection is closed and bounded hence compact.

If we choose $e_j \in E_j$ then there is a subsequence that converges to $e_0 \in E_1$. But that subsequence also converges in $E_2$ to $e_0$. In fact it converges in each $E_j$ to $e_0$.

So $e_0 \in \bigcap_{j=1}^{\infty} E_j$.

14. (10 points) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Prove that $f^{-1}((-1, 1)) = \{x \in \mathbb{R} : f(x) \in (-1, 1)\}$ is an open set.

Let $x \in f^{-1}((-1, 1))$. Since $f$ is continuous at $x$ if $\epsilon > 0$ then there is a $\delta > 0$ such that $|x - t| < \delta \Rightarrow |f(x) - f(t)| < \epsilon$. If we choose $\epsilon$ so that $(f(x) - \delta, f(x) + \delta) \subset (-1, 1)$, then this says that $(x - \delta, x + \delta) \subset f^{-1}((-1, 1))$.

Hence $f^{-1}((-1, 1))$ is open.
15. (10 points) Let $U_\alpha$ be open sets in $\mathbb{R}$ for $\alpha$ ranging over an index set $A$. Prove that $\bigcup_{\alpha \in A} U_\alpha$ is open.

Let $x \in \bigcup_{\alpha \in A} U_\alpha$. Then $x \in U_{\alpha_0}$ for some $\alpha_0$.

So there is $\varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subset U_{\alpha_0} \subset \bigcup_{\alpha \in A} U_\alpha$.

Hence $\bigcup_{\alpha \in A} U_\alpha$ is open.

16. (10 points) Find all real and complex roots of the polynomial equation

$$p(x) = x^3 - 3x^2 + 4x - 2.$$  

Clearly $p(1) = 0$.

\[
\begin{align*}
&1 - 3 + 4 - 2 \\
&= 0
\end{align*}
\]

\[
\begin{array}{c}
| -2x^2 + 4x \\
-2x^2 + 4x \\
-2x^2 + 2x \\
-2x - 2
\end{array}
\]

We solve $x^2 - 2x + 2$ by

\[
x = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i
\]

So the roots are $1$, $1+i$, $1-i$.  

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17. (10 points) Find all complex cube roots of the complex number \( z = -\sqrt{3} + 1i. \)

\[
\begin{align*}
\zeta &= 2e^{i\frac{5\pi}{6}} \\
\rho^3 e^{i\phi/3} &= 2e^{i\frac{5\pi}{6}} \Rightarrow r = 2^{1/3}, \phi = \frac{5\pi}{18} \\
\rho^3 e^{i\theta/3} &= 2e^{i\frac{17\pi}{6}} \Rightarrow r = 2^{1/3}, \theta = \frac{17\pi}{18} \\
\rho^3 e^{i\psi/3} &= 2e^{i\frac{29\pi}{6}} \Rightarrow r = 2^{1/3}, \psi = \frac{29\pi}{18}
\end{align*}
\]

\[\sum \rho_0 \text{ root } \phi_0 = 2^{1/3} e^{i\frac{5\pi}{18}}, 2^{1/3} e^{i\frac{17\pi}{18}}, 2^{1/3} e^{i\frac{29\pi}{18}}\]

18. (10 points) What is the definition of a Cauchy sequence of rational numbers? What do Cauchy sequences have to do with the construction of the real numbers?

A sequence \( \{a_j\} \subset \mathbb{Q} \) is Cauchy if, given \( \varepsilon > 0 \), there is an \( N > 0 \) s.t.

\[ j, k > N \Rightarrow |a_j - a_k| < \varepsilon. \]

We say that two Cauchy sequences of rational numbers are related if, given \( \varepsilon > 0 \), \( \exists N > 0 \) s.t.

\[ j, k > N \Rightarrow |a_j - b_k| < \varepsilon. \]

The equivalence classes are a model for the real numbers \( \mathbb{R} \).
19. (10 points) Prove by example that if \( A, B, C \) are closed sets in the real numbers \( \mathbb{R} \) then \((A \cup B) \setminus C\) is not necessarily closed or open.

Let \( A = [0, 2], B = [1, 3], C = [2, 4] \).

Then \((A \cup B) \setminus C = [0, 2]\).

This is neither open nor closed.

20. (10 points) If \( G, H \) are groups then \( G \times H \) is a group. Prove this statement. Say explicitly what the binary operation is, and check the three axioms for a group. If \( G \) and \( H \) are both abelian, then is \( G \times H \) abelian?

Define

\((g, h) \cdot (g', h') = (g \cdot g', h \cdot h')\).

Clearly \( G \times H \) is closed under the group law.

1) \((g, h) \cdot [(g', h'), (g'', h'')] = (g, h) - [(g' \cdot g'', h' \cdot h'')] \]

\[= [(g \cdot g'', h \cdot h'')] - [(g, h) \cdot (g', h'')] \]

\[= [(g \cdot g', h \cdot h')] - [(g', h')] \]

\[= [(g, h) \cdot (g', h')] \cdot (g', h') \]

b) Let \( e = (e_G, e_H) \). Then \( e \cdot (g, h) = (g, h) \cdot e = (g, h) \).

c) If \((g, h) \in G \times H\) then \((g^{-1}, h^{-1})\) is the multiplicative inverse.