MATH 310

SOLUTIONS TO HW 4

pp. 119-127

1. reflexive: If m is an integer, then $m+m=2m$ is even.

symmetric: If $m+n$ is even then $n+m$ is even.

transitive: If $m+n$ is even and $n+k$ is even then $(m+n)+(n+k)$ is even so $(m+k)+2n$ is even hence $m+k$ is even.

The equivalence classes are all even numbers and all odd numbers.

3. reflexive: If $(m,n) \in \mathbb{Z} \times \mathbb{Z}$, then $m+n = m+n$ so $(m,n) \sim (m,n)$.

symmetric: If $(m,n) \sim (k,l)$ then $m+l = n+k$ so $k+n = l+m$ hence $(k,l) \sim (m,n)$.

transitive: If $(m,n) \sim (k,l)$, then $m+l = n+k$. If $(k,l) \sim (s,t)$, then $k+t = l+s$. Adding these equations together gives

$$m + \cancel{l} + \cancel{k} + t = n + \cancel{k} + \cancel{l} + s$$

$$m + t = n + s.$$

So $(m,n) \sim (s,t)$.

The equivalence class containing (m,n) also contains $(\underbrace{m-n}_{m-n \,\in\, \mathbb{Z}}, 0)$. So the collection of equivalence classes is just a model for $\mathbb{Z}$.

$S = \{a, b, c, d\}$, $T = \{1, 2, 3, 4, 5, 6, 7\}$

9b) $\{(a,5), (c,4), (d,3)\}$

<u>Not</u> a function because b is not in the domain.

d) $\{(a,2), (b,2), (c,3), (d,3)\}$.

This <u>is</u> a function.

f) $\{(d,7), (c,6), (c,5), (a,4), (b,1)\}$.

<u>Not</u> a function because c is mapped to two different places.

10 b) $g : \mathbb{Z} \to \mathbb{Z}$, $g(m) = 2m - 7$

<u>Is</u> one-to one because $g(m) = g(m')$

$$\Rightarrow 2m - 7 = 2m' - 7$$
$$\Rightarrow m = m'.$$

<u>Is</u> onto because if $n \in \mathbb{Z}$ then $g\left(\frac{n+7}{2}\right) = n$.

d) $f : \mathbb{Q} \to \mathbb{Q}$, $f(x) = x^2 + 4$

Not one-to-one because $f(-1) = f(1) = 5$.

Not onto because $\nexists\, x$ such that $f(x) = -10$.

f) $h: \mathbb{R} \to \mathbb{R}$, $h(n) = \sqrt{n^2+1}$.

Not one-to-one because $h(-1) = h(1) = \sqrt{2}$.

Not onto because $\nexists x$ s.t. $h(x) = -5$.

h) $g: \mathbb{N} \to \mathbb{Z}$, $g(k) = k^3 + 2k$

This is one-to-one because g is strictly increasing.

Not onto because $\nexists x$ with $g(x) = -6$.

11. This is certainly a partial ordering. It is the way that we order words in a dictionary.

NOTES

17. $f: \mathbb{N} \to \mathbb{Z}$

$$f(n) = \begin{cases} -\frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

18. $f: \mathbb{Q} \to \mathbb{N}$

$$f(q) = \begin{cases} q & \text{if } q \in \mathbb{N} \\ 0 & \text{if } q \in \mathbb{Q} \setminus \mathbb{N} \end{cases}$$

19. If there were a function $g: \mathbb{N} \to \mathbb{R}$ that is onto, then define a new function $h: \mathbb{R} \to \mathbb{N}$ such that if $x \in \mathbb{R}$, then $h(x)$ is some element of $g^{-1}(x)$. Then h is one-to-one.

So $\text{card}(\mathbb{R}) \leq \text{card}(\mathbb{N})$;

That is impossible.

20 a) $\mathbb{N} \times \mathbb{Q}$ is the product of countable sets so countable.

c) $\mathbb{R} \times \mathbb{Q}$

$f: \mathbb{R} \to \mathbb{R} \times \mathbb{Q}$

$x \mapsto (x, 0)$

SUMMARY

is one-to-one. So $\text{card}(\mathbb{R}) \leq \text{card}(\mathbb{R} \times \mathbb{Q})$

Hence $\mathbb{R} \times \mathbb{Q}$ is uncountable.

f) $\mathbb{R} \setminus \mathbb{N}$. $\mathbb{N}$ is countable. If $\mathbb{R} \setminus \mathbb{N}$ were countable, then $\mathbb{R} = \mathbb{N} \cup (\mathbb{R} \setminus \mathbb{N})$ would be countable — which is false. So $\mathbb{R} \setminus \mathbb{N}$ is uncountable.

h) This is isomorphic to the set of all sequences of 0s and 1s. So uncountable.

i) The set S_k of all seqs. of 3s and 7s with k elements is countable. The set described is

$$\bigcup_{k \in \mathbb{N}} S_k$$

hence countable.

l) The set of all triples of natural numbers is countable. The set of all quadruples of natural numbers is countable, Etcetera. The union of these is countable.

24. Represent each real number as a decimal. Replace any infinitely repeating decimal like $.9999\ldots 99\ldots$ with its shorter version 1. Then map

$$a_1 a_2 a_3 . a_4 a_5 a_6 a_7 \cdots \text{ to}$$

$$(a_1 a_3 . a_5 a_7 \cdots , a_2 . a_4 a_6 \cdots).$$

Call this function f.

DATE:

PURPOSE:

CUE COLUMN

NOTES If $(a,b), (c,d) \in \mathbb{R}\times\mathbb{R}$, examine

$A = f^{-1}(a,b)$

$B = f^{-1}(c,d)$.

If $A < B$ then say $(a,b) < (c,d)$.

If $A > B$ then say $(a,b) > (c,d)$.

28. $x, y \in \mathbb{R}$ are related if $x-y$ is rational.

reflexive: $x - x = 0 \in \mathbb{Q}$ so $x \sim x$

symmetric: If $x \sim y$ then $x-y$ is rational so $y-x$ is rational hence $y \sim x$.

transitive: If $x \sim y$ and $y \sim z$ then $x-y$ is rational and $y-z$ is rational. So

$x - z = (x-y) + (y-z)$ is rational.

Hence $x \sim z$.

Thus this is an equivalence relation

If $x \in \mathbb{R}$, then the equivalence class

$[x] = \{x + q : q \in \mathbb{Q}\}$ so is countable.

$\mathbb{R}$ is the union of these equiv. classes.

If there were only ctbly-many equiv. classes then $\mathbb{R}$ would be countable. So there are

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uncountably many equiv. classes.

CUE COLUMN NOTES

34. $f : X \to \mathbb{Z}$ is surjective.

Construct a function $h : \mathbb{Z} \to X$ by

$z \in \mathbb{Z}$ goes to an element of $f^{-1}(z)$.

Then h is one-to-one. So

$$\text{card}(\mathbb{Z}) \le \text{card}(X).$$

Hence X is at least countable, so infinite.

42. Let $S_j = \{x \in \mathbb{R} : 0 < x < \frac{1}{j}\}$, $j = 1, 2, \ldots$

Then $S_1 \supset S_2 \supset \cdots$ and each S_j is uncountable. But

$$\bigcap_j S_j = \emptyset.$$

47. $\mathbb{N} \times \mathbb{N} = \bigcup_{j=1}^{\infty} \{(m,n) : 1 \le m \le j,\ 1 \le n \le j\}$

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