SOLUTIONS

Math 310 Fall, 2020
Krantz

FIRST MIDTERM EXAM

General Instructions: Read the statement of each problem carefully. If you want full credit on a problem then you must show your work. If you only write the answer then you will *not* receive full credit.

Be sure to ask questions if anything is unclear. This exam has 8 questions and is worth 100 points. You will have 50 minutes to take this exam.

(8 points) **1.** Write the converse and the contrapositive of the sentence

If dogs can fly, then chickens have lips.

Label each one.

Converse: If chickens have lips, then dogs can fly.

Contrapositive: If chickens do not have lips, then dogs cannot fly.

(8 points) **2.** Are the statements ($\mathbf{A} \Rightarrow \sim \mathbf{B}$) and ($\sim \mathbf{A} \vee \sim \mathbf{B}$) logically equivalent? Why or why not?

A	B	$\sim A$	$\sim B$	$A \Rightarrow \sim B$	$\sim A \vee \sim B$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

The columns for $A \Rightarrow \sim B$ and $\sim A \vee \sim B$ are identical. So the statements are logically equivalent.

(10 points) **3.** Express the statement $\exists x, \sim \mathbf{P}(x)$ using $\forall$ instead of $\exists$.

$\sim \forall x, P(x)$.

(8 points) 4. Prove that the product of an even integer and an odd integer is even.

Let $k = 2n$ be even and $l = 2m+1$ be odd.

Then $k \cdot l = 2n(2m+1) = 2\cdot(2mn+n)$ is even.

(10 points) 5. Prove that the integer 5 does not have a rational square root.

Suppose to the contrary that 5 does have a rational square root m/n in lowest terms. Then $5 = (m/n)^2$ so $5 = m^2/n^2$ or $5n^2 = m^2$.

Now 5 divides the left side so 5 divides the right side. Since 5 is prime, 5 divides m. So $m = 5k$, some positive integer k. Thus $5n^2 = (5k)^2 = 25k^2$ so $n^2 = 5k^2$. We see that 5 divides the right side so 5 divides the left side. Since 5 is prime, 5 divides n.

We conclude that 5 divides m and 5 divides n. So m/n is not in lowest terms. Contradiction. Hence 5 does not have a rational square root.

(8 points) **6.** Use mathematical induction to prove that

$$1+2+\cdots+n=\frac{n(n+1)}{2}$$

for any positive integer n.

a) $P(n)$ is $1+2+\cdots+n=\frac{n(n+1)}{2}$

b) $P(1)$ is $1=\frac{1\cdot(1+1)}{2}$ which is true.

c) Assume $P(n)$:

$$1+2+\cdots+n=\frac{n(n+1)}{2}$$

Add $(n+1)$ to both sides:

$$1+2+\cdots+n+(n+1)=\frac{n(n+1)}{2}+n+1$$

$$=\frac{n^2+n+2n+2}{2}=\frac{n^2+3n+2}{2}$$

$$=\frac{(n+1)(n+2)}{2}.$$

This is $P(n+1)$. Induction complete.

(8 points) **7.** Give a truth table for the statement $(\mathbf{A}\wedge\sim\mathbf{B})\Rightarrow(\sim\mathbf{A}\vee\mathbf{B})$.

A	B	$\sim A$	$\sim B$	$A\wedge\sim B$	$\sim A\vee B$	$(A\wedge\sim B)\Rightarrow(\sim A\vee B)$
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(8 points) **8.** Use any method to prove the pigeon-hole principle.

Suppose not. Then we place $n+1$ letters in n mailboxes and no box contains two letters.

Thus the total number of letters is <u>at most</u>

$$\underbrace{1+1+1+\cdots+1+1}_{n \text{ times}}$$

Hence the total number of letters is at most n. Contradiction. So some box contains two letters.

(8 points) **9.** Let S and T be sets. Prove that

$$(S \setminus T) \cup (T \setminus S) = (S \cup T) \setminus (S \cap T).$$

Let $x \in (S \setminus T) \cup (T \setminus S)$, So $x \in S \setminus T$ or $x \in T \setminus S$. Thus either x is in S but not T or x is in T but not S. We conclude that x is in S or T, but x is not in both S and T. So $x \in (S \cup T) \setminus (S \cap T)$

Conversely, let $x \in (S \cup T) \setminus (S \cap T)$. So x is in S or T but not in both S and T. Hence either $x \in S \setminus T$ or $x \in T \setminus S$. We conclude that $x \in (S \setminus T) \cup (T \setminus S)$.

Putting together the two inclusions gives

$$(S \setminus T) \cup (T \setminus S) = (S \cup T) \setminus (S \cap T).$$

(9 points) **10.** Let $S = \{1,2,3,4,5\}$, $T = \{3,4,5,7,8,9\}$, $U = \{1,2,3,4,9\}$. Calculate

(a) $(S \cap T) \cup U$

(b) $(S \cap U) \cup T$

(c) $(S \cup T) \setminus (S \cap T)$

(a) $S \cap T = \{3,4,5\}$. $(S \cap T) \cup U = \{1,2,3,4,5,9\}$.

(b) $S \cap U = \{1,2,3,4\}$.

$(S \cap U) \cup T = \{1,2,3,4,5,7,8,9\}$.

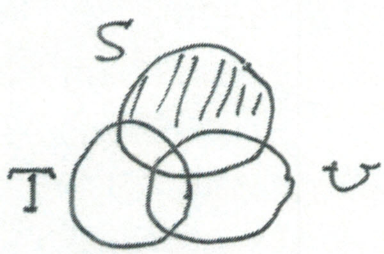
(c) $S \cup T = \{1,2,3,4,5,7,8,9\}$

$S \cap T = \{3,4,5\}$

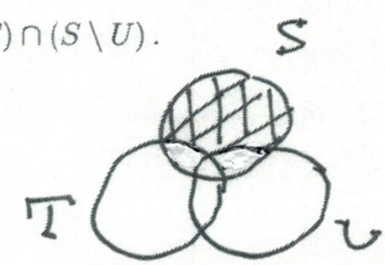
$(S \cup T) \setminus (S \cap T) = \{1,2,7,8,9\}$

(8 points) **11.** Let S, T, and U be sets. Draw two Venn diagrams to illustrate the identity.

$$S \setminus (T \cup U) = (S \setminus T) \cap (S \setminus U).$$



$S \setminus (T \cup U)$



$(S \setminus T) \cap (S \setminus U)$

(7 points) **12.** Give an explicit description of the power set of $A = \{1, 2, a, b\}$.

$\{\{1\}, \{2\}, \{a\}, \{b\}, \{1,2\}, \{1,a\}, \{1,b\},$
$\{2,a\}, \{2,b\}, \{a,b\}, \{1,2,a\}, \{1,2,b\}, \{2,a,b\},$
$\{1,a,b\}, \{1,2,a,b\}, \emptyset\}$