PRACTICE EXAM FOR SECOND MIDTERM

(8 points) 1. Let $S = \{3, 4, 6\}$ and $T = \{3, 5\}$. What is $S \cup T$? What is $S \cap T$? What is $S \setminus T$?

\[
S \cup T = \{3, 4, 5, 6\}
\]
\[
S \cap T = \{3\}
\]
\[
S \setminus T = \{4, 6\}
\]
2. Let \( S = \{2, 4\} \) and \( T = \{a, b, d\} \). What is \( S \times T \)? What is \( T \times S \)?

\[
S \times T = \{(2, a), (2, b), (2, d), (4, a), (4, b), (4, d)\}
\]

\[
T \times S = \{(2, 2), (2, 4), (b, 2), (b, 4), (d, 2), (d, 4)\}
\]

3. Draw two Venn diagrams to illustrate the identity

\[
(T \cup U) \setminus S = (T \setminus S) \cup (U \setminus S).
\]
4. What is the power set of \{\lambda, A, 2\}?

\{\{\lambda\}, \{A\}, \{2\}, \{\lambda, A\}, \{\lambda, 2\}, \{A, 2\}, \{\lambda, A, 2\}, \emptyset\}
5. Which of these functions is one-to-one? Which is onto (give a brief reason for each answer)?

(a) \( f : \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = x^2 + x \)

\[ f(0) = f(-1) = 0 \] so not one-to-one.

Quadratic formula \( \Rightarrow \) \( \exists x \) with \( f(x) = -5 \), not onto.

(b) \( g : \mathbb{N} \rightarrow \mathbb{N} \quad g(n) = n(n + 3) \)

\( g \) is increasing so one-to-one.

\( \exists n \) with \( g(n) = 1 \)

(c) \( h : \mathbb{R} \rightarrow \mathbb{R} \quad h(x) = x \cos x \)

\( h \) is continuous & takes arbitrarily large positive values and arbitrarily large negative values, so is onto.

A horizontal line will intersect the graph more than once so not one-to-one.

6. Which of these sets is countable and which uncountable (give a brief reason for each answer)?
(a) \( C \times C \)
\[ f : C \rightarrow C \times C \]
\[ z \mapsto (z, 0) \text{ is one-to-one,} \]
So \( \text{card}(C) \leq \text{card}(C \times C) \), hence \( C \times C \) uncountable.

(b) \( \mathbb{Z} \times \mathbb{Q} \)
\( \mathbb{Z} \) is countable, \( \mathbb{Q} \) is countable,
So \( \mathbb{Z} \times \mathbb{Q} \) is countable

(c) \( \mathbb{N} \times \mathbb{R} \)
\[ g : \mathbb{R} \rightarrow \mathbb{N} \times \mathbb{R} \]
\[ x \mapsto (0, x) \text{ is one-to-one,} \]
So \( \text{card}(\mathbb{R}) \leq \text{card}(\mathbb{N} \times \mathbb{R}) \),
Hence \( \mathbb{N} \times \mathbb{R} \) uncountable.

(10 points) 7. Calculate the inverse of the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) given by

\[ f(x) = \begin{cases} 
  x^3 & \text{if } x \leq 0 \\
  x & \text{if } x > 0.
\end{cases} \]
(14 points) 8. Prove that the collection of $S$ of rational numbers with denominator 7 is countable.

$$f: S \to \mathbb{Z}$$

$$\frac{m}{7} \rightarrow m$$

is one-to-one and onto. So $S$ is countable.
(10 points) 9. Explain why the product of an uncountable set and an uncountable set is uncountable.

Let \( S \) be uncountable. Then, fixing \( s_0 \in S \),

\[ g : S \to S \times S \]

\[ s \mapsto (s, s_0) \] is one-to-one.

Thus \( \text{card} \ (S) \leq \text{card} \ (S \times S) \).

Hence \( S \times S \) is uncountable.

(8 points) 10. Explain why the union of a uncountable set and an uncountable set is uncountable.

Let \( S \) be uncountable

\( T \) be uncountable

Then \( f : S \to S \cup T \)

\[ s \mapsto s \]

is one-to-one.

So \( \text{card} \ (S) \leq \text{card} \ (S \cup T) \).

Hence \( S \cup T \) is uncountable.