Math 318
Krantz

Spring, 2013
May 7, 2013

Final Exam

General Instructions: Read the statement of each problem carefully. Do only what is requested—nothing more and nothing less. Provide a *complete solution* to each problem.

If you only write the answer then you will not get full credit. If you need extra room for your work then use the backs of the pages.

Be sure to ask questions if anything is unclear.

(10 points) **1.** If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at a point $\mathbf{a}$, then the entries of the matrix $Df(\mathbf{a})$ are the partial derivatives of f. Explain why this is true.

Apply the definition of differentiability

$$\lim_{\mathbf{h} \to 0} \frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - Df(\mathbf{a})\mathbf{h}}{\|\mathbf{h}\|} = 0$$

with $\mathbf{h} = t\mathbf{e}_j$ ($\mathbf{e}_j$ a coordinate vector).

(10 points) **2.** If f and g are real-valued functions on an open set $U \subset \mathbb{R}^n$ and if $D_{\mathbf{v}}f(\mathbf{a}) = D_{\mathbf{v}}g(\mathbf{a})$ for every point $\mathbf{a} \in U$ and every unit vector $\mathbf{v}$, then show that f and g differ by a constant.

We assume that $D_{\underline{v}}(f-g)(\underline{a}) = 0$.

So $[\nabla(f-g)(a)] \cdot \underline{v} = 0 \quad \forall \underline{a}, \underline{v}$.

Hence $\nabla(f-g) \equiv 0$

so $f-g \equiv$ constant.

(10 points) **3.** If

$$\mathbf{f}(x, y) = \begin{pmatrix} x^2 - y^2 \\ x^2 + y^2 \end{pmatrix}$$

and

$$\mathbf{g}(x, y) = x^3 - 4y^2 ,$$

then use the Chain Rule to calculate the derivative at the origin of $\mathbf{g} \circ \mathbf{f}$.

$$D(\underline{g} \circ \underline{f})(0) = D\underline{g}(\underline{f}(0)) \cdot D\underline{f}(0) \quad (*)$$

Now $D\underline{g} = (3x^2 \;\; -8y)$

, $D\underline{f} = \begin{pmatrix} 2x & -2y \\ 2x & 2y \end{pmatrix}$

So $(*) = \left[D\underline{g}\begin{pmatrix}0\\0\end{pmatrix}\right] \cdot \left[D\underline{f}\begin{pmatrix}0\\0\end{pmatrix}\right]$

$$= (0 \;\; 0) \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = (0 \;\; 0).$$

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(10 points) **4.** If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping then define the quantity $\|T\|$. Prove that

$$\|T\mathbf{x}\| \leq \|T\|\|\mathbf{x}\|$$

for any $\mathbf{x} \in \mathbb{R}^n$.

$$\|T\| = \max_{\|x\|=1} \|Tx\|.$$

For any $x \in \mathbb{R}^n$, let $\tilde{x} = \frac{x}{\|x\|}$. Then

$\|T\tilde{x}\| \leq \|T\|$ so $\left\|T\frac{x}{\|x\|}\right\| \leq \|T\|$

hence $\|Tx\| \leq \|T\| \|x\|$.

(10 points) **5.** Let V be a vector space. What is the definition of the *dimension* of V? Give an example of a vector space of dimension 4. Give an example of an infinite-dimensional vector space.

Any basis has the same number of elements.
If V has a finite basis, then the number of elements in that basis is the dimension.

Let $V =$ all polynomials of degree ≤ 3.
Then $\{1, x, x^2, x^3\}$ is a basis. Dimension is 4.

Let $V' =$ all continuous fcns. on $[0, 1]$.
This has ∞ dimension.

(10 points) **6.** Use Gaussian elimination to completely solve the linear system

$$\begin{array}{ccccccc} x_1 & & -3x_3 & +2x_4 & = & 1 \\ x_1 & -x_2 & -2x_3 & +x_4 & = & 0 \\ & x_2 & +x_3 & -x_4 & = & 2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 1 \\ 1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 1 \\ 0 & -1 & 1 & -1 & -1 \\ 0 & 1 & 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 2 & -2 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1/2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 5/2 \\ 0 & 1 & 0 & 0 & 3/2 \\ 0 & 0 & 1 & -1 & 1/2 \end{array}\right]$$

$x_4 = t$
$x_3 = t + 1/2$
$x_2 = 3/2$
$x_1 = t + 5/2$

$$\underset{\sim}{x} = \begin{pmatrix} 5/2 \\ 3/2 \\ 1/2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

(10 points) **7.** Use Gaussian elimination to find the inverse of the matrix

$$\begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 2/3 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2/3 & -1/3 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & -3/2 \end{array}\right]$$

So inverse is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1/2 & 0 & -3/2 \end{bmatrix}$

(10 points) **8.** Calculate the curvature κ of the curve $\mathbf{g} : \mathbb{R} \longrightarrow \mathbb{R}^3$ given by

$$\mathbf{g}(t) = \begin{pmatrix} \sin t \\ \cos t \\ t \end{pmatrix}$$

at the point $\mathbf{g}(0) = (0, 1, 0)$.

Reparametrize as $\tilde{g}(s) = \begin{pmatrix} \sin s/\sqrt{2} \\ \cos s/\sqrt{2} \\ s/\sqrt{2} \end{pmatrix}$.

Then $T(s) = \begin{pmatrix} \frac{1}{\sqrt{2}} \cos s/\sqrt{2} \\ -\frac{1}{\sqrt{2}} \sin s/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, $T'(s) = \begin{pmatrix} -\frac{1}{2} \sin s/\sqrt{2} \\ -\frac{1}{2} \cos s/\sqrt{2} \\ 0 \end{pmatrix}$

$\kappa(s) = \|T'(s)\| = \sqrt{\left(-\frac{1}{2}\sin\frac{s}{\sqrt{2}}\right)^2 + \left(-\frac{1}{2}\cos\frac{s}{\sqrt{2}}\right)^2 + 0^2} = \frac{1}{2}$.

(10 points) **9.** Let f be a real-valued function of two variables that is differentiable at all points. Explain why, at any given point, the gradient gives the direction of greatest increase.

$D_{\underline{v}} f(\underline{a}) = \underline{v} \cdot \nabla f(\underline{a})$ so $|D_{\underline{v}} f(\underline{a})| \leq \|v\| \, \|\nabla f(\underline{a})\|$

by Cauchy - Schwarz

and this is $= \|\nabla f(\underline{a})\|$.

So the greatest directional derivative has size $\|\nabla f(\underline{a})\|$.

That occurs in the direction $\dfrac{\nabla f(\underline{a})}{\|\nabla f(\underline{a})\|}$.

(10 points) **10.** What is the equation of the tangent line to the curve $\mathbf{g}(t) = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + t^2\mathbf{k}$ at the point $(1, 0, 0)$?

$$g'(t) = -\sin t\, \underline{i} + \cos t\, \underline{j} + 2t\, \underline{k}$$

$$g'(0) = 0\, \underline{i} + 1\, \underline{j} + 0\, \underline{k}.$$

So the line is

$$(1, 0, 0) + t\langle 0, 1, 0\rangle$$

$$x = 1$$
$$y = t$$
$$z = 0$$

(10 points) **11.** Let V be the vector space consisting of all polynomials of one real variable with degree less than or equal to 4. Give an explicit basis for V. What is the dimension of V?

basis $= \{1, x, x^2, x^3, x^4\}$

dimension $= 5$.

(10 points) **12.** Consider the vectors

$$\begin{aligned} \mathbf{v}_1 &= \langle 0, 1, 1, 0 \rangle \\ \mathbf{v}_2 &= \langle 1, 0, 1, 0 \rangle \\ \mathbf{v}_3 &= \langle 1, 1, 1, 1 \rangle \\ \mathbf{v}_4 &= \langle 0, 0, 1, -1 \rangle \end{aligned}$$

in $\mathbb{R}^4$. What is the span of these vectors? Give an explicit geometric desciption of this span.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The span is 3-dimensional — the span of the first three rows of this last matrix.

(10 points) **13.** Determine whether the vectors

$$\begin{aligned} \mathbf{v}_1 &= \langle 0, 1, 1, 1 \rangle \\ \mathbf{v}_2 &= \langle 1, 1, 1, 0 \rangle \\ \mathbf{v}_3 &= \langle -2, -1, -1, 1 \rangle \end{aligned}$$

$1\underline{v}_1 - 2\underline{v}_2 = \langle -2, -1, -1, 1 \rangle = \underline{v}_3$,

So <u>not</u> linealy indepentent.

are linearly independent in $\mathbb{R}^4$.

(10 points) **14.** Prove that, if $S \subseteq \mathbb{R}^n$ is compact and if $\mathbf{a}$ is a point not in S then there is a point $\mathbf{x}_0 \in S$ such that

$$\|\mathbf{x}_0 - \mathbf{a}\| = \text{g.l.b.}\{\|\mathbf{x} - \mathbf{a}\| : x \in S\}.$$

The function $f : S \to \mathbb{R}$ given by $f(\underline{x}) = \|\underline{x} - \underline{a}\|$ is continuous. So, on a compact set, it has a minimum value $\delta > 0$. And there must exist a point $\underline{x}_0$ where f takes that minimum value.

(10 points) **15.** Find and identify all critical points of the function $f(x,y) = x^2 - y^3 + 3x^2y$.

$$\nabla f = (2x + 6xy, -3y^2 + 3x^2).$$

$$2x + 6xy = 0$$
$$-3y^2 + 3x^2 = 0 \Rightarrow x = \pm y \Rightarrow y = 0, \quad y = -1/3$$

$$\overset{a}{\overset{\shortparallel}{(-1/3, -1/3)}}, \overset{b}{\overset{\shortparallel}{(0,0)}}, \overset{c}{\overset{\shortparallel}{(1/3, -1/3)}} \text{ are critical points}$$

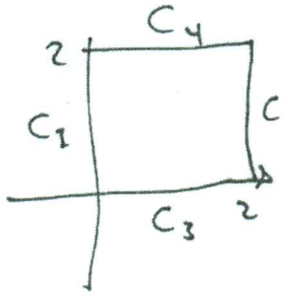
$$\text{Hessian} = \begin{pmatrix} 2+6y & 6x \\ 6x & -6y \end{pmatrix}$$

$\mathcal{H}_{f,\underline{a}} = 2h_2^2 - 4h_1h_2$ indefinite saddle

$\mathcal{H}_{f,\underline{b}} = 0$ semidefinite no conclusion

$\mathcal{H}_{f,\underline{c}} = 2h_2^2 + 4h_1h_2$ indefinite saddle

(10 points) **16.** Find the global maximum and global minimum of the function $f(x,y) = x^2 - xy$ on the set $[0,2] \times [0,2]$.



$$\nabla f = (2x - y \quad -x)$$

$$\left.\begin{array}{l} 2x - y = 0 \\ -x = 0 \end{array}\right\} \begin{array}{l} x = 0 \\ y = 0 \end{array}$$

Only critical point is $(0,0)$.

$$\text{Hessian} = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$$

$\mathcal{H}_{f,(0,0)} = 2h_1^2 - 2h_1h_2$ indefinite saddle

So extrema are on edges.

on C_1 $f \equiv 0$ on C_2 $f(1,t) = 1 - t \;\Big\langle\; \begin{matrix} -1 \\ 1 \end{matrix}$

on C_3 $f(t,0) = t^2 \;\Big\langle\; \begin{matrix} 4 \\ 0 \end{matrix}$ on C_4 $f(t,2) = t^2 - 2t \;\Big\langle\; \begin{matrix} 0 \\ -1 \end{matrix}$

So max value is 4 at $(2,0)$

min value is -1 at $(1,2)$.

(10 points) **17.** Give the precise definition of a positive definite quadratic form.

A form Q is positive definite if $Q(\underline{x}) > 0$ for all $\underline{x} \neq 0$.

(10 points) **18.** Explain why the mapping

$$f(x, y) = (y/2 + 1, x/3 + 2)$$

has a unique fixed point.

Notice that

$$\| f(x,y) - f(x',y') \| = \| (y/2 + 1, x/3 + 2) - (y'/2 + 1, x'/3 + 2) \|$$

$$= \| ((y-y')/2, (x-x')/3 \|$$

$$= \sqrt{\frac{(x-x')^2}{9} + \frac{(y-y')^2}{4}} \leq \frac{1}{2}\sqrt{(x-x')^2 + (y-y')^2}$$

$$= \frac{1}{2} \| (x,y) - (x',y') \|.$$

Thus f is a contraction mapping. By Banach's theorem, f has a fixed point.

(10 points) **19.** Define $f(x,y) = (xy, x^2 - y^2)$. Explain why f is invertible in a neighborhood of $(1,1)$.

$$\operatorname{Jac} f = \begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix}$$

$\det \operatorname{Jac} f = -2y^2 - 2x^2$.

$\det \operatorname{Jac} f(1,1) = -4 \neq 0$.

By the inverse function theorem, f is invertible in a neighborhood of $(1,1)$.

(10 points) **20.** Consider the curve $x^2 + y^2 = 1$. Near what points of this curve can we solve for y in terms of x? Near what points can we not do so? Give specific reasons in each case.

Let $h(x,y) = x^2 + y^2$.

Then $\frac{\partial h}{\partial y} = 2y$.

If $y \neq 0$, then $\partial h/\partial y \neq 0$ and the implicit function theorem applies. So we can solve for y in terms of x.

If $y = 0$ then $\partial h/\partial y = 0$ and we cannot.