Project:

Math 318 Krantz ①

Solutions to HW2

Notes: §1.4

1. b) $2A - B = 2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$

$= \begin{pmatrix} 0 & 3 \\ 2 & 5 \end{pmatrix}$

d) $C + D$ makes no sense because the dimensions do not match.

f) $BA = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} 5 & 8 \\ 13 & 20 \end{pmatrix}.$

h) CA makes no sense because the dimensions do not match.

To Do:

2a) If $A\underline{x} = 0$ for all $\underline{x} \in \mathbb{R}^n$, then take $\underline{x} = \langle 1, 0, \ldots, 0 \rangle$. Then

$0 = A\underline{x} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, So $a_{11} = 0, a_{21} = 0, \ldots, a_{m1} = 0$.

Taking $\underline{x} = \underline{e}_2$ gives $a_{12}, a_{22}, \ldots, a_{m2} = 0$

etc.

So $A = 0$.

Project:

Notes:

3. Clearly $A0 = 0$.

If $\underline{u}, \underline{v}$ are n-vectors with $A\underline{u} = A\underline{v} = 0$ then $A(\underline{u}+\underline{v}) = 0$ so $\underline{u}+\underline{v}$ is in the subspace.

If c is a scalar then

$$A(c\underline{u}) = cA\underline{u} = 0.$$

Thus V is a subspace.

7. $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

If $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then

$$A_\theta \underline{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\theta)x_1 - (\sin\theta)x_2 \\ (\sin\theta)x_1 + (\cos\theta)x_2 \end{bmatrix}$$

So $\|A_\theta \underline{x}\| = \sqrt{[(\cos\theta)x_1 - (\sin\theta)x_2]^2 + [(\sin\theta)x_1 + (\cos\theta)x_2]^2}$

$= \sqrt{(\cos^2\theta + \sin^2\theta)x_1^2 + (\sin^2\theta + \cos^2\theta)x_2^2}$

$= \sqrt{x_1^2 + x_2^2} = \|\underline{x}\|$.

To Do:

Project:

Notes:

The angle ψ between $\underline{x}$ and $A_\theta \underline{x}$ is given by

$$\cos\psi = \frac{\underline{x} \cdot A_\theta \underline{x}}{\|\underline{x}\| \, \|A_\theta \underline{x}\|} = \frac{\langle x_1, x_2 \rangle \cdot \langle \cos\theta x_1 - \sin\theta x_2, \sin\theta x_1 + \cos\theta x_2 \rangle}{\|x\|^2}$$

$$= \frac{(\cos\theta) x_1^2 - \sin\theta x_1 x_2 + \sin\theta x_1 x_2 + \cos\theta x_2^2}{\|\underline{x}\|^2}$$

$$= \frac{\cos\theta \, \|x\|^2}{\|\underline{x}\|^2} = \cos\theta$$

11. $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

Clearly $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$.

$$A^2 = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{bmatrix} \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{bmatrix} = \begin{bmatrix} d_1^2 & & \\ & \ddots & \\ & & d_m^2 \end{bmatrix}$$

$$A^k = \begin{bmatrix} d_1^k & & \\ & \ddots & \\ & & d_m^k \end{bmatrix}.$$

To Do:

Project:

Notes:

13, a) T has matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

S has matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b) $T \circ S$ has matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

c) $S \circ T$ has matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

To Do:

18, If $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ then $A \neq 0$

but $A^2 = 0$,

If $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ then $A^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

but $A^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Project:

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Notes:

To Do:

25. $(A^TA)^T = A^T A^{TT} = A^TA$.

§ 1.5

4a) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$= \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}$$

$$\det(AB) = (a\alpha + b\gamma)(c\beta + d\delta) - (c\alpha + d\gamma)(a\beta + b\delta)$$

On the other hand,

$$(\det A)(\det B) = (ad - bc)(\alpha\delta - \beta\gamma)$$

One observes that these are the same.

5. a) $\underline{x} \times \underline{y} = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{pmatrix}$

$$= \underline{i}\,2 - \underline{j}\,2 + \underline{k}\,2$$

b) $\underline{x} \times \underline{y} = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & 1 \\ 7 & 1 & -5 \end{pmatrix}$

$$= \underline{i}\,9 - \underline{j}\,(-12) + \underline{k}\,15.$$

Project:

Notes:

6b) Let $\underline{u} = \overrightarrow{AB} = \langle 1, 0, -1 \rangle$

$\underline{v} = \overrightarrow{AC} = \langle 1, 2, 1 \rangle$

Area of triangle $= \frac{1}{2} \| \underline{u} \times \underline{v} \|$

$$= \frac{1}{2} \left\| \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{pmatrix} \right\|$$

$$= \frac{1}{2} \| [\underline{i} 2 - \underline{j} 2 + \underline{k} 2] \|$$

$$= \frac{1}{2} \sqrt{4 + 4 + 4}$$

$$= \sqrt{3}.$$

d) Let $\underline{u} = \overrightarrow{AB} = \langle 1, -2, 1 \rangle$

$\underline{v} = \overrightarrow{AC} = \langle 7, 1, -5 \rangle.$

Area of triangle $= \frac{1}{2} \| \underline{u} \times \underline{v} \|$

$$= \frac{1}{2} \left\| \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & 1 \\ 7 & 1 & -5 \end{pmatrix} \right\|$$

$$= \frac{1}{2} \| \underline{i} 9 - \underline{j}(-12) + \underline{k} 15 \|$$

$$= \frac{1}{2} \sqrt{81 + 144 + 225} \quad = \frac{1}{2} \sqrt{450} = \frac{3}{2} \sqrt{50}.$$

To Do:

Project:

⑦

Notes:

7 b) Let $\underline{u} = \overrightarrow{AB} = \langle 1, -2, 1 \rangle$

$\underline{v} = \overrightarrow{AC} = \langle 7, 1, -5 \rangle$.

Then

$$\underline{u} \times \underline{v} = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & 1 \\ 7 & 1 & -5 \end{pmatrix}$$

$$= \underline{i}\,9 - \underline{j}(-12) + \underline{k}\,10.$$

If $\underline{X}$ is a point on the plane then

$$\overrightarrow{XA} \cdot \langle 9, 12, 10 \rangle = 0$$

$$\langle x-0, y-0, z-0 \rangle \cdot \langle 9, 12, 10 \rangle = 0$$

$$9x + 12y + 10z = 0$$

d) Let $\underline{u} = \overrightarrow{AB} = \langle 1, -2, 1 \rangle$

$\underline{v} = \overrightarrow{AC} = \langle 7, 1, -5 \rangle$.

Then

$$\underline{u} \times \underline{v} = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & 1 \\ 7 & 1 & -5 \end{pmatrix}$$

$$= \underline{i}\,9 - \underline{j}(-12) + \underline{k}\,10.$$

If $\underline{X}$ is any point on the plane then

$$\overrightarrow{XA} \cdot \langle 9, 12, 10 \rangle = 0.$$

To Do:

Project:

Notes:

9. $x_1 + x_2 - 2x_3 = 0$

$2x_1 + x_2 + x_3 = 0$.

let $x_1 = t$

$$\begin{aligned} t + x_2 - 2x_3 &= 0 \\ 2t + x_2 + x_3 &= 0 \\ \hline -t \quad - 3x_3 &= 0 \end{aligned}$$

$$x_3 = -t/3$$

$$\therefore \; x_2 = -\frac{5}{3}t$$

So the intersection is the line

$$x_1 = t$$
$$x_2 = -\frac{5}{3}t$$
$$x_3 = -\frac{1}{3}t$$

To Do:

14, a) $\underline{x} \times \underline{y} = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$

$\underline{y} \times \underline{x} = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{pmatrix}$

Switching rows introduces a minus sign, so

$\underline{x} \times \underline{y} = - \underline{y} \times \underline{x}$

Project:

Notes:

$\underline{x} \times (\underline{y} + \underline{z}) = \underline{x} \times \underline{y} + \underline{x} \times \underline{z}$

follows from distributivity properties of the determinant

b) Let $\underline{x} = \langle 1, 1, 0 \rangle$, $\underline{y} = \langle 0, 1, 1 \rangle$, $\underline{z} = \langle 1, 0, 1 \rangle$.

Then

$$\underline{x} \times \underline{y} = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \underline{i} - \underline{j} + \underline{k}$$

$$(\underline{x} \times \underline{y}) \times \underline{z} = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \underline{i}(-1) - \underline{j} \cdot 0 + \underline{k}(1)$$

$$\underline{y} \times \underline{z} = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \underline{i} - \underline{j}(-1) + \underline{k}(-1)$$

$$\underline{x} \times (\underline{y} \times \underline{z}) = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \underline{i}(-1) - \underline{j}(-1) + \underline{k}(-1)$$

We see that

$(\underline{x} \times \underline{y}) \times \underline{z} \neq \underline{x} \times (\underline{y} \times \underline{z})$.

To Do: