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Exercises 2.3

1. If $\underline{l}$ and $\underline{m}$ are two distinct putative limits, then let $\varepsilon = \|\underline{l} - \underline{m}\|/2$. Then $\exists\, \delta_1 > 0$ s.t. if $\|\underline{x} - \underline{a}\| < \delta_1$ then $\|\underline{f}(\underline{x}) - \underline{l}\| < \varepsilon$. Also $\exists\, \delta_2 > 0$ s.t. if $\|\underline{x} - \underline{a}\| < \delta_2$ then $\|\underline{f}(\underline{x}) - m\| < \varepsilon$. Let $\delta = \min(\delta_1, \delta_2)$. If $\|\underline{x} - \underline{a}\| < \delta$ then

$$\begin{aligned}\|l - m\| &\le \|l - \underline{f}(\underline{x})\| + \|\underline{f}(\underline{x}) - m\| \\ &< \varepsilon + \varepsilon \\ &= \frac{\|l - m\|}{2} + \frac{\|l - m\|}{2}.\end{aligned}$$

So $\|l - m\| < \|l - m\|$. Contradiction. So $\underline{l} = \underline{m}$.

2. Let $\varepsilon > 0$. Select $\delta = \varepsilon$. If $\|\underline{x} - \underline{a}\| < \delta$ then

$$\begin{aligned}|f(\underline{x}) - f(\underline{a})| &= \big|\, \|\underline{x}\| - \|\underline{a}\| \,\big| \\ &\le \|\underline{x} - \underline{a}\| < \delta = \varepsilon.\end{aligned}$$

So f is continuous at $\underline{a}$.

7. a) A linear map $\mathcal{L}$ is given by matrix multiplication:

$$\mathcal{L}\underline{x} = M\underline{x} \qquad M = (m_{ij}).$$

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Each element of the vector Mx is given by the dot product of a row of M with $\underline{x}$.
We know from Example 1 that this is continuous.
So $\mathcal{L}$ is continuous.

b) The inequality is just Schwarz's inequality.

Then $\|A\underline{x} - A\underline{a}\| = \|A(\underline{x} - \underline{a})\|$

$$\le \Big(\sum_{i,j} a_{ij}^2\Big)^{1/2} \|\underline{x} - \underline{a}\|.$$

If $\varepsilon > 0$, let $\delta = \dfrac{\varepsilon}{\left(\sum_{i,j} a_{ij}^2\right)^{1/2}}$. Then

$$\|\underline{x} - \underline{a}\| < \delta \Rightarrow \|A\underline{x} - A\underline{a}\| < \varepsilon.$$

8. a) When $|x| < \frac{1}{10}$, $|y| < \frac{1}{10}$ then $|x+y+1| > 8/10$.

So $\left|\dfrac{xy}{1+x+y}\right| < \dfrac{10}{8}|xy| \to 0$.

b) We know that $\lim_{t\to 0} \dfrac{\sin t}{t} = 1$. So

$$\lim_{(x,y)\to \underline{0}} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1.$$

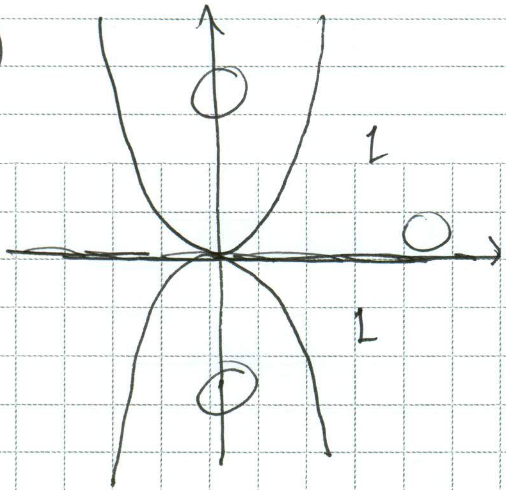
c) $f\begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{x^2 - y^2}{x - y} = \dfrac{(x-y)(x+y)}{x-y} = x + y \to 0$

e) $x^2 + y^2 \to 0$ so $-1/(x^2+y^2) \to -\infty$ so

$e^{-1/x^2+y^2} \to 0$.

13. Let E be closed. Then ${}^cE$ is open. So $f^{-1}({}^cE)$ is open. But $f^{-1}({}^cE) = {}^c(f^{-1}(E))$. So $f^{-1}(E)$ is closed.

15. a)



If $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ vertically, then $\lim f\begin{pmatrix} x \\ y \end{pmatrix} = 0$

If $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ along the curve $y = x^2/2$, then $\lim f\begin{pmatrix} x \\ y \end{pmatrix} = 1$.

But if $y = \lambda t$ is any line terminating at 0, then the line eventually gets inside the parabola. So the limit along the line is 0.

b) Let

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} 0 & \text{if } |y| > x^4 \text{ or } y = 0 \\ 1 & \text{otherwise} \end{cases}$$

An analysis similar to part (a) gives the result.

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Exercises 3.1

1. a) $\frac{\partial f}{\partial x} = 3x^2 + 3y^2$, $\frac{\partial f}{\partial y} = 6xy - 2$

c) $\frac{\partial f}{\partial x} = \frac{1}{1+(y/x)^2} \cdot \frac{-y}{x^2}$, $\frac{\partial f}{\partial y} = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x}$

e) $\frac{\partial f}{\partial x} = 1 \cdot \log x + (x+y^2) \cdot \frac{1}{x}$, $\frac{\partial f}{\partial y} = 2y \log x$

2. b) $D_{\underline{\underline{v}}} f = \langle 2x+y, x \rangle \cdot \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$

$= \sqrt{2}x + y/\sqrt{2} - x/\sqrt{2}$. At $\underline{\underline{a}} = (2,1)$ this is $\sqrt{2} + 1/\sqrt{2}$

d) $D_{\underline{\underline{v}}} f = \langle -ye^{-x}, e^{-x} \rangle \cdot \langle 3/5, 4/5 \rangle$

$= -\frac{3}{5} y e^{-x} + \frac{4}{5} e^{-x}$. At $\underline{\underline{a}} = (0,1)$ this is $1/5$.

3. b) direction of greatest increase is the gradient, which is $\langle -ye^{-x}, e^{-x} \rangle$. Evaluated at $\underline{\underline{a}}$ this is $\langle -1, 1 \rangle$. Normalized to length 1, the answer is $\underline{\underline{v}} = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$.

7. $\frac{\partial g}{\partial x} = f'\left(\frac{x}{y}\right) \cdot \frac{1}{y}$ $\quad \frac{\partial g}{\partial y} = f'(x/y) \cdot \frac{-x}{y^2}$

$x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = x f'\left(\frac{x}{y}\right) \cdot \frac{1}{y} + y f'\left(\frac{x}{y}\right) \cdot \frac{-x}{y^2} = 0.$

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9. $\frac{\partial f}{\partial x}(\underline{0}) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$ with $x = 0, y = 0$

$= \lim_{h \to 0} \frac{0 - 0}{h} = 0$

$\frac{\partial f}{\partial y}(\underline{0}) = \lim_{k \to 0} \frac{f(0, 0+k) - f(0,0)}{k}$

$= \lim_{k \to 0} \frac{0 - 0}{k} = 0.$

$D_{\underline{v}} f(\underline{0}) = \lim_{t \to 0} \frac{f((0,0) + t\underline{v}) - f(0,0)}{t}$

$= \lim_{t \to 0} \frac{(t v_1)(t v_2)}{t} = \lim_{t \to 0} t v_1 v_2 = 0.$

10. a) $f(x,y) = \frac{x^2 y}{x^4 + y^2}$

$D_{\underline{v}} f(\underline{0}) = \lim_{t \to 0} \frac{f((0,0) + t\underline{v}) - f(0,0)}{t}$

$= \lim_{t \to 0} \frac{\frac{(tv_1)^2 tv_2}{(tv_1)^4 + (tv_2)^2} - 0}{t}$

$= \lim_{t \to 0} \frac{v_1 v_2}{t^2 v_1^4 + v_2^2} = \frac{v_1 v_2}{v_2^2} = \frac{v_1}{v_2}$