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Math 318 Homework 5 Solutions

Section 3.2

1\. b) The tangent plane is the graph of

$$\underline{g}(\underline{x}) = \underline{f}(\underline{a}) + D\underline{f}(\underline{a})(\underline{x}-\underline{a}).$$

For us,

$$\underline{f}(\underline{a}) = (-1)^2 + 2^2 = 5$$

$$D\underline{f} = (2x, 2y), \quad D\underline{f}(\underline{a}) = (-2, 4)$$

$$D\underline{f}(\underline{a})(\underline{x}-\underline{a}) = (-2,4)\begin{pmatrix} x_1+1 \\ x_2-2 \end{pmatrix} = -2(x_1+1) + 4(x_2-2).$$

So $g(\underline{x}) = 5 + (-2)(x_1+1) + 4(x_2-2)$.

We write this as

$$z = 5 + (-2)(x+1) + 4(y-2)$$

or

$$2x - 4y + z = -5.$$

d) $f(\underline{a}) = \sqrt{4-1^2-1^2} = \sqrt{2}$

$$D\underline{f} = \left(\frac{-x}{\sqrt{4-x^2-y^2}}, \frac{-y}{\sqrt{4-x^2-y^2}} \right), \quad D\underline{f}(\underline{a}) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$D\underline{f}(\underline{a})(\underline{x}-\underline{a}) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}$$

The tangent plane is

$$z = \sqrt{2} + -\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y + \sqrt{2}$$

or $\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y + z = 2\sqrt{2}$

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(2)

2. b) $D_{\underline{v}} f(\underline{a}) = \underline{v} \cdot \nabla f(\underline{a})$

$= \langle 1, -1 \rangle \cdot \langle e^x \cos y, -e^x \sin y \rangle \Big|_{(0, \pi/4)}$

$= \langle 1, -1 \rangle \cdot \langle 1 \cdot \sqrt{2}/2, -1 \cdot \sqrt{2}/2 \rangle$

$= \sqrt{2}/2 + \frac{\sqrt{2}}{2} = \sqrt{2}.$

d) $D_{\underline{v}} f(\underline{a}) = \underline{v} \cdot \nabla f(\underline{a})$

$= \langle 2/\sqrt{5}, 1/\sqrt{5} \rangle \cdot \langle 2x, 2y \rangle \Big|_{(2,1)}$

$= \langle 2/\sqrt{5}, 1/\sqrt{5} \rangle \cdot \langle 4, 2 \rangle$

$= 8/\sqrt{5} + 2/\sqrt{5} = \frac{10}{\sqrt{5}} = 2\sqrt{5}.$

3. b) $D\underline{f} = \begin{bmatrix} -\sin t \\ \cos t \\ e^t \end{bmatrix}$

d) $D\underline{f} = \begin{bmatrix} yz & xz & xy \\ 1 & 1 & 2z \end{bmatrix}$

e) $$\underline{Df} = \begin{bmatrix} \cos y & -x\sin y \\ \sin y & x\cos y \\ 0 & 1 \end{bmatrix}$$

7. Let $\underline{\underline{L}} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. So L is given by an $m \times n$ matrix A. We claim that $D\underline{\underline{L}}(\underline{\underline{a}}) = \underline{\underline{L}}$ for any $\underline{\underline{a}}$.

Now

$$\lim_{\underline{\underline{h}} \to 0} \frac{\underline{\underline{L}}(\underline{\underline{a}}+\underline{\underline{h}}) - \underline{\underline{L}}(\underline{\underline{a}}) - D\underline{\underline{L}}(\underline{\underline{a}})\underline{\underline{h}}}{\|\underline{\underline{h}}\|}$$

$$= \lim_{\underline{\underline{h}} \to 0} \frac{A(\underline{\underline{a}}+\underline{\underline{h}}) - A\underline{\underline{a}} - A\underline{\underline{h}}}{\|\underline{\underline{h}}\|}$$

$$= \lim_{\underline{\underline{h}} \to 0} \frac{\underline{\underline{0}}}{\|\underline{\underline{h}}\|} = \underline{\underline{0}}.$$

So $A = D\underline{\underline{L}}(\underline{\underline{a}})$ for any $\underline{\underline{a}}$.

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10. $$\underline{\underline{Df}} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

$$\underline{\underline{Df}}(\underline{\underline{a}}) = \begin{pmatrix} 2a_1 & -2a_2 \\ 2a_2 & 2a_1 \end{pmatrix}$$

$$= 2\sqrt{a_1^2 + a_2^2} \begin{pmatrix} \frac{a_1}{\sqrt{a_1^2+a_2^2}} & \frac{-a_2}{\sqrt{a_1^2+a_2^2}} \\ \frac{a_2}{\sqrt{a_1^2+a_2^2}} & \frac{a_1}{\sqrt{a_1^2+a_2^2}} \end{pmatrix}$$

This has the form of a rotation as discussed in the text. Also note that the rows are unit vectors and orthogonal to each other. And the determinant is 1.

15. Consider the function $\varphi_{\underline{\underline{h}}}(t) = f(\underline{\underline{a}} + t\underline{\underline{h}})$ for $\underline{\underline{h}}$ a unit vector and $t \in (-1, 1)$.
Then $\varphi_{\underline{\underline{h}}}$ is a differentiable fn. of one real variable and $\varphi_{\underline{\underline{h}}}(0) \geq \varphi_{h}(t)$ for $|t|$ small. By Fermat's test, $\varphi_h'(0) = 0$. Writing out the deriv. of $f(\underline{\underline{a}} + t\underline{\underline{h}})$ gives the result.

Section 3.3

1. $(f \circ \underline{g})'(0) = D\underline{f}(\underline{g}(0)) \cdot D\underline{g}(0)$

Now

$$D\underline{g} = \begin{pmatrix} -\sin t + \cos t \\ 1 \\ 2t + 4 \end{pmatrix}$$

$$D\underline{g}(0) = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$D\underline{f}(\underline{g}(0)) = D\underline{f}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}$$

So

$$(f \circ g)'(0) = D\underline{f}(\underline{g}(0)) \cdot D\underline{g}(0)$$

$$= \begin{bmatrix} 2 & 1 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = 2 + 1 - 4 = -1.$$

2. $D(\underline{f} \circ \underline{g})(\underline{0}) = D\underline{f}(\underline{g}(\underline{0})) \cdot D\underline{g}(\underline{0})$.

Now $D\underline{f} = \begin{bmatrix} -\cos x & 2 \\ e^{x+3y} & 3e^{x+3y} \\ y & x + 3y^2 \end{bmatrix}$, $\underline{g}(\underline{0}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

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$$\underline{\underline{Df}}(\underline{\underline{g}}(\underline{\underline{0}})) = \begin{bmatrix} -1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\underline{Dg}} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & z & y \end{pmatrix}, \quad \underline{\underline{Dg}}(\underline{\underline{0}}) = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\underline{\underline{Df}}(\underline{\underline{g}}(\underline{\underline{0}})) \cdot \underline{\underline{Dg}}(\underline{\underline{0}}) = \begin{bmatrix} -1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 1 \\ 6 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Also $D(\underline{\underline{g}} \circ \underline{\underline{f}})(\underline{\underline{0}}) = D\underline{\underline{g}}(\underline{\underline{f}}(\underline{\underline{0}})) \cdot D\underline{\underline{f}}(\underline{\underline{0}})$.

$$\underline{\underline{f}}(\underline{\underline{0}}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{\underline{Dg}}(\underline{\underline{f}}(\underline{\underline{0}})) = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

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$$D\underline{\underline{f}}(\underline{\underline{0}}) = \begin{bmatrix} -1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}.$$

So $D\underline{\underline{g}}(\underline{\underline{f}}(\underline{\underline{0}})) \cdot D\underline{\underline{f}}(\underline{\underline{0}})$

$$= \begin{pmatrix} 3 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 \\ 1 & 3 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 9 \\ -1 & 2 \end{pmatrix}.$$

3. $(f \circ \underline{\underline{g}})'(t) = Df(\underline{\underline{g}}(t)) \cdot D\underline{\underline{g}}(t)$.

Now $Df = (2x+2, 2y, 2z)$

$$D\underline{\underline{g}} = \begin{pmatrix} -\sin t \\ \cos t \\ \cos(\frac{t}{2}) \end{pmatrix}$$

$$Df(\underline{\underline{g}}(t)) = (2\cos t + 2, 2\sin t, 4\sin(t/2))$$

$$(f \circ \underline{\underline{g}})'(t) = Df(\underline{\underline{g}}(t)) \cdot D\underline{\underline{g}}(t)$$

$$= (2\cos t + 2, 2\sin t, 4\sin(t/2)) \cdot \begin{pmatrix} -\sin t \\ \cos t \\ \cos(\frac{t}{2}) \end{pmatrix}$$

$$= -2\cos t \sin t - 2\sin t + 2\sin t \cos t + 4\sin(\tfrac{t}{2})\cos(\tfrac{t}{2})$$

$$= -2\sin t + 2\sin t = 0.$$

So f is constant on the curve described by $\underline{\underline{g}}$.

8. We predict that the derivative of $\frac{1}{\underline{g}}$ at $\underline{a}$ will be

$$\frac{-D\underline{g}(\underline{a})}{\underline{g}^2(\underline{a})}.$$

So we examine

$$\lim_{\underline{h}\to \underline{0}} \frac{\frac{1}{\underline{g}(\underline{a}+\underline{h})} - \frac{1}{\underline{g}(\underline{a})} - \left[\frac{-D\underline{g}(\underline{a})}{\underline{g}^2(\underline{a})}\right]}{\|\underline{h}\|}$$

$$= \lim_{\underline{h}\to \underline{0}} \frac{\frac{\underline{g}(\underline{a})-\underline{g}(\underline{a}+\underline{h})}{\underline{g}(\underline{a})\underline{g}(\underline{a}+\underline{h})} + \left[\frac{D\underline{g}(\underline{a})}{\underline{g}^2(\underline{a})}\right]\underline{h}}{\|\underline{h}\|}$$

$$= \lim_{\underline{h}\to \underline{0}} \frac{\frac{\underline{g}(\underline{a})-\underline{g}(\underline{a}+\underline{h})}{\underline{g}(\underline{a})\underline{g}(\underline{a}+\underline{h})} - \frac{\underline{g}(\underline{a})-\underline{g}(\underline{a}+\underline{h})}{\underline{g}^2(\underline{a})}}{\|\underline{h}\|}$$

$$+ \lim_{\underline{h}\to \underline{0}} \frac{\frac{\underline{g}(\underline{a})-\underline{g}(\underline{a}+\underline{h})}{\underline{g}^2(\underline{a})} + \frac{D\underline{g}(\underline{a})}{\underline{g}^2(\underline{a})}\underline{h}}{\|\underline{h}\|}$$

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$$= \lim_{\underline{h} \to \underline{0}} \frac{\left[g(\underline{a}) - g(\underline{a}+\underline{h})\right]\left[\frac{1}{g(\underline{a})g(\underline{a}+\underline{h})} - \frac{1}{g^2(\underline{a})}\right]}{\|\underline{h}\|}$$

$$+ \frac{1}{g^2(\underline{a})} \lim_{\underline{h} \to \underline{0}} \frac{g(\underline{a}) - g(\underline{a}+\underline{h}) + Dg(\underline{a})\underline{h}}{\|\underline{h}\|}$$

Now the fraction in the second summand tends to 0 so the entire expression tends to 0.

In the first expression,

$$\left[g(\underline{a}) - g(\underline{a}+\underline{h})\right] \sim Dg(\underline{a})\underline{h}$$

and the expression

$$\left[\frac{1}{g(\underline{a})g(\underline{a}+\underline{h})} - \frac{1}{g(\underline{a})^2}\right] \to 0,$$

So the product divided by $\|\underline{h}\|$ tends to $\underline{0}$.

That completes the argument.

11. We have

$$f(t\underline{x}) = t^k f(\underline{x}).$$

Differentiating both sides with respect to t, and using the Chain Rule, gives

$$Df(t\underline{x})\cdot\underline{x} = kt^{k-1}f(\underline{x}). \qquad (*)$$

Setting $t=1$ now gives the result.

For the converse direction, fix an $\underline{x}$ and define

$$g(\lambda) = f(\lambda\underline{x}) - \lambda^k f(\underline{x}).$$

Note that $g(1)=0$. For $\lambda>0$,

$$g'(\lambda) = Df(\lambda\underline{x})\underline{x} - k\lambda^{k-1}f(\underline{x})$$

$$= \lambda^{-1}\left[Df(\lambda\underline{x})\lambda\underline{x}\right] - k\lambda^{k-1}f(\underline{x})$$

$$= \lambda^{-1}kf(\lambda\underline{x}) - k\lambda^{k-1}f(\underline{x})$$

by the case of $(*)$ when $t=1$.

Thus $\lambda g'(\lambda) = k[f(\lambda\underline{x}) - \lambda^k f(\underline{x})]$

$$= kg(\lambda).$$

So g satisfies the differential equation

$$g'(\lambda) - \frac{k}{\lambda}g(\lambda) = 0, \quad g(1)=0.$$

The only solution to this equation is $g\equiv 0$.

So $f(\lambda\underline{x}) = \lambda^k f(\underline{x})$.

14. Write

$$F(t) = \int_0^{v(t)} h(s)\,ds - \int_0^{u(t)} h(s)\,ds$$

Define

$$G(r) = \int_0^r h(s)\,ds.$$

Then $F(t) = G(v(t)) - G(u(t))$.

So $F'(t) = G'(v(t)) \cdot v'(t) - G'(u(t)) \cdot u'(t)$

$$= h(v(t)) \cdot v'(t) - h(u(t)) \cdot v'(t).$$