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Section 3.4

1 b) By implicit differentiation,

$$3y^2 + 3x \cdot 2y\frac{dy}{dx} + ye^{xy} + xe^{xy}\frac{dy}{dx} - \pi\cos(\pi y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-3y^2 - ye^{xy}}{6xy + xe^{xy} - \pi\cos(\pi y)}$$

Substitute in $x=0,\ y=1$

$$\frac{dy}{dx} = \frac{-3-1}{0+0+\pi} = \frac{4}{\pi}$$

This is the slope of the tangent line. The equation is

$$y - 1 = \frac{4}{\pi}(x-0).$$

2 b) gradient $= \langle 2ye^{xy}z^5,\ z^2 + 2xe^{xy}z^5,\ 10e^{xy}z^4 \rangle$

At the point $(0,2,1)$ this is

gradient $= \langle 4, 1, 10 \rangle$.

Thus the tangent plane is

$$\langle 4, 2, 10 \rangle \cdot \langle x-0, y-2, z-1 \rangle = 0$$

$$4x + 1y + 10z = 12$$

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d) gradient $= \langle 2e^{2x+z}\cos(3y) - y, e^{2x+z}(-3\sin 3y) - x,$

$e^{2x+z}\cos(3y) + 1 \rangle.$

At the point $(-1, 0, 2)$ this is

gradient $= \langle 2, 1, 2 \rangle$

Thus the plane is

$$\langle 2, 1, 2 \rangle \cdot \langle x-(-1), y-0, z-2 \rangle = 0$$

$$2x + y + 2z = 2.$$

4, a) Direction of greatest ascent is $\langle 3, 4, -1 \rangle$.

b) If the flow is in the e_2 direction then one unit of motion gives vertical descent of -4.

So we need the angle between

$\langle 0, 1, -4 \rangle$ and $(0, 1, 0)$ [horizontal]

This is given by

$$\cos\theta = \frac{\langle 0, 1, -4 \rangle \cdot \langle 0, 1, 0 \rangle}{\sqrt{0^2+1^2+4^2}\,\sqrt{0^2+1^2+0^2}} = \frac{1}{\sqrt{17}}.$$

6. We know that $f \circ \underline{g} = c$.

Taking the derivative,

$$Df(\underline{g}(x)) \cdot D\underline{\underline{g}}(x) = 0.$$

∇f. tangent vector to C.

10. Consider $g(t) = f(t, -2t)$

Then $g'(t) = \frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} \equiv 0$.

So g is constant.

So g takes the same value at any pt. of the form $(t, -2t)$.

Hence we can define

$F(2x+y) = f(x,y)$ unambiguously.

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Section 3.5

1. $\underline{\underline{g}}(t) \cdot \underline{\underline{g}}'(t) = 0$

So $\frac{d}{dt}\left(\underline{\underline{g}}(t) \cdot \underline{\underline{g}}(t)\right) = 0$.

Hence $\underline{\underline{g}}(t) \cdot \underline{\underline{g}}(t) \equiv c$.

Thus $\underline{\underline{g}}(t)$ lies on a sphere of radius $\sqrt{c}$.

3. $\underline{\underline{f}} \cdot \underline{\underline{g}} = c$

$\underline{\underline{f}}' \cdot \underline{\underline{g}} + \underline{\underline{f}} \cdot \underline{\underline{g}}' = 0$

$\underline{\underline{f}}' \cdot \underline{\underline{g}} = - \underline{\underline{g}}' \cdot \underline{\underline{f}}$.

If $\underline{\underline{f}}$ and $\underline{\underline{g}}$ are unit vectors, this says that

The projection of the tangent to $\underline{\underline{f}}$ onto the position of $\underline{\underline{g}}$ equals the projection of the tangent to $\underline{\underline{g}}$ onto the position of $\underline{\underline{f}}$.

5. Set $\underline{\underline{g}}(t) = (g_1(t), \ldots, g_n(t))$. So our equation is

$$g_j'(t) = \lambda(t)\, g_j(t) \quad , \quad j = 1, \ldots, n.$$

Let $\Lambda(t)$ be an antiderivative of λ.

Then $g_j(t) = C_j e^{\Lambda(t)}$, each j.

Then it is clear that $\underline{\underline{g}}/\|\underline{\underline{g}}\|$ is constant.

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7 b) $\displaystyle\int_{-1}^{1}\sqrt{\left(\tfrac{1}{2}(e^{t}-e^{-t})\right)^{2}+\left(\tfrac{1}{2}(e^{t}+e^{-t})\right)^{2}+1^{2}}\,dt$

$$=\int_{-1}^{1}\sqrt{\tfrac{1}{4}e^{2t}+\tfrac{1}{4}e^{-2t}+\tfrac{1}{4}e^{2t}+\tfrac{1}{4}e^{-2t}+1}\,dt$$

$$=\int_{-1}^{1}\sqrt{\left(\tfrac{1}{\sqrt{2}}e^{t}+\tfrac{1}{\sqrt{2}}e^{-t}\right)^{2}}\,dt$$

$$=\int_{-1}^{1}\tfrac{1}{\sqrt{2}}e^{t}+\tfrac{1}{\sqrt{2}}e^{-t}\,dt=\tfrac{1}{\sqrt{2}}e^{t}-\tfrac{1}{\sqrt{2}}e^{-t}\Big]_{-1}^{1}$$

$$=\tfrac{1}{\sqrt{2}}e-\tfrac{1}{\sqrt{2}}\tfrac{1}{e}-\tfrac{1}{\sqrt{2}}\tfrac{1}{e}+\tfrac{1}{\sqrt{2}}e$$

$$=\sqrt{2}e-\sqrt{2}/e.$$

d) $\displaystyle\int_{0}^{2\pi}\sqrt{[a(1-\cos t)]^{2}+[a(\sin t)]^{2}}\,dt$

$$=\int_{0}^{2\pi}\sqrt{a^{2}(1-2\cos t+\cos^{2}t+\sin^{2}t)}\,dt$$

$$=\int_{0}^{2\pi}2\sqrt{a}\cdot\sqrt{1-2\cos t}\,dt$$

This problem would work better if

$$\underline{g}(t)=\begin{bmatrix}a(1-\sin t)\\ a(1-\cos t)\end{bmatrix}.$$

9. $\mathbf{v}(t) = v(t)\,\mathbf{T}(t)$

$\mathbf{a}(t) = v'(t)\,\mathbf{T}(t) + v(t)\,\mathbf{T}'(t)$

But

$$\mathbf{T}'(t) = \|\mathbf{T}'(t)\|\,\mathbf{N}(t) = v(t)\,\kappa(t)\,\mathbf{N}(t)$$

So $\mathbf{a}(t) = v'(t)\,\mathbf{T}(t) + v(t)^2\,\kappa(t)\,\mathbf{N}(t)$.

Hence

$$\mathbf{v}(t) \times \mathbf{a}(t) = v(t)\,\mathbf{T}(t) \times \left[v'(t)\,\mathbf{T}(t) + v(t)^2\kappa(t)\,\mathbf{N}(t)\right]$$

$$= v(t)\,\mathbf{T}(t) \times \left[v(t)^2\kappa(t)\,\mathbf{N}(t)\right]$$

$$= v(t)^3\kappa(t)\,\mathbf{T}(t) \times \mathbf{N}(t)$$

Hence $\|\mathbf{v}(t) \times \mathbf{a}(t)\| = v(t)^3\,\kappa(t)$.

$$\kappa(t) = \frac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{v(t)^3}.$$

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Section 3.6

1. a) $$\frac{\partial f}{\partial x}(0,y) = \lim_{h\to 0} \frac{f(0+h,y) - f(0,y)}{h}$$

$$= \lim_{h\to 0} \frac{hy\frac{h^2-y^2}{h^2+y^2} - 0}{h}$$

$$= \lim_{h\to 0} y\frac{h^2-y^2}{h^2+y^2} = -y.$$

$$\frac{\partial f}{\partial y}(x,0) = \lim_{k\to 0} \frac{f(x,0+k) - f(x,0)}{k}$$

$$= \lim_{k\to 0} \frac{xk\frac{x^2-k^2}{x^2+k^2} - 0}{k}$$

$$= \lim_{k\to 0} x\frac{x^2-k^2}{x^2+k^2} = x$$

b) $$\frac{\partial f}{\partial x}(0,0) = 0$$

$$\frac{\partial f}{\partial y\partial x}(0) = \lim_{k\to 0} \frac{\frac{\partial f}{\partial x}(0,0+k) - \frac{\partial f}{\partial x}(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{-k - 0}{k} = -1$$

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$$\frac{\partial f}{\partial x \partial y}(0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(0+h,0) - \frac{\partial f}{\partial y}(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h-0}{h} = +1.$$

c) If f were C^2 at 0 then the derivatives would commute. But they don't.

2. b) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[\log(x^2+y^2)\right]$

$$= \frac{\partial}{\partial x}\left[\frac{1}{x^2+y^2}\cdot 2x\right] + \frac{\partial}{\partial y}\left[\frac{1}{x^2+y^2}\cdot 2y\right]$$

$$= \frac{(x^2+y^2)\cdot 2 - 2x\cdot 2x}{(x^2+y^2)^2} + \frac{(x^2+y^2)\cdot 2 - 2y\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{2y^2 - 2x^2 + 2x^2 - 2y^2}{(x^2+y^2)^2} = 0.$$

d) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(x^2+y^2+z^2)^{-1/2}$

$$= \frac{\partial}{\partial x}\left(-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x\right) + \frac{\partial}{\partial y}\left(-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2y\right)$$

$$+ \frac{\partial}{\partial z}\left(-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2z\right)$$

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(9)

$$= \frac{3}{4}(x^2+y^2+z^2)^{-5/2} \cdot 4x^2 - \frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2$$

$$+ \frac{3}{4}(x^2+y^2+z^2)^{-5/2} \cdot 4y^2 - \frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2$$

$$+ \frac{3}{4}(x^2+y^2+z^2)^{-5/2} \cdot 4z^2 - \frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2$$

$$= (x^2+y^2+z^2)^{-5/2}\Big[3x^2+3y^2+3z^2 - (x^2+y^2+z^2) - (x^2+y^2+z^2) - (x^2+y^2+z^2)\Big]$$

$$= 0.$$

3 b) $\frac{\partial f}{\partial t} = -5c\sin 5x \sin 5ct$

$$\frac{\partial^2 f}{\partial t^2} = -25c^2 \sin 5x \cos 5ct$$

$$\frac{\partial f}{\partial x} = 5\cos 5x \cos 5ct$$

$$\frac{\partial^2 f}{\partial x^2} = -25\sin 5x \cos 5ct$$

Clearly $\frac{\partial^2 f}{\partial x^2} = +\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$.

That is the heat equation.

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6. $$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial g_1}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial g_2}{\partial v}$$

$$\frac{\partial}{\partial u}\left(\frac{\partial F}{\partial v}\right) = \frac{\partial^2 f}{\partial x^2}\frac{\partial g_1}{\partial u}\frac{\partial g_1}{\partial v} + \frac{\partial f}{\partial x}\frac{\partial^2 g_1}{\partial u \partial v}$$

$$+ \frac{\partial^2 f}{\partial x \partial y}\frac{\partial g_2}{\partial u}\frac{\partial g_1}{\partial v}$$

$$+ \frac{\partial^2 f}{\partial x \partial y}\frac{\partial g_2}{\partial u}\frac{\partial g_2}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial^2 g_2}{\partial u \partial v}$$

$$+ \frac{\partial^2 f}{\partial y^2}\frac{\partial g_2}{\partial u}\frac{\partial g_2}{\partial v}$$

7. $$\frac{\partial F}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

$$\frac{\partial^2 F}{\partial r^2} = \frac{\partial^2 f}{\partial x^2}\cos^2\theta + \frac{\partial^2 f}{\partial x \partial y}\cos\theta\sin\theta + \frac{\partial^2 f}{\partial y \partial x}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta$$

$$\frac{\partial F}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos(\theta))$$

$$\frac{\partial^2 F}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2} r^2\sin^2\theta + \frac{\partial f}{\partial x}(-r\cos\theta) + \frac{\partial^2 f}{\partial x \partial y}(-r\sin\theta)(r\cos\theta)$$

$$+ \frac{\partial^2 f}{\partial y \partial x}(r\cos\theta)(-r\sin\theta) + \frac{\partial^2 f}{\partial y^2}(r\cos\theta)(r\cos\theta)$$

$$+ \frac{\partial f}{\partial y}(-r\sin\theta)$$

Then

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}$$

$$= \underline{\frac{\partial^2 f}{\partial x^2}\cos^2\theta} + \cancel{2\frac{\partial^2 f}{\partial x \partial y}\cos\theta\sin\theta} + \underline{\frac{\partial^2 f}{\partial y^2}\sin^2\theta}$$

$$+ \cancel{\frac{1}{r}\frac{\partial f}{\partial x}\cos\theta} + \cancel{\frac{1}{r}\frac{\partial f}{\partial y}\sin\theta}$$

$$+ \underline{\frac{\partial^2 f}{\partial x^2}\sin^2\theta} + \cancel{2\frac{\partial^2 f}{\partial x \partial y}(-\sin\theta\cos\theta)} + \underline{\frac{\partial^2 f}{\partial y^2}\cos^2\theta}$$

$$- \cancel{\frac{1}{r}\frac{\partial f}{\partial x}\cos\theta} - \cancel{\frac{1}{r}\frac{\partial f}{\partial y}\sin\theta}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

10 a) The equation is

$$\left(1+\left(\frac{\partial f}{\partial x}\right)^2\right)\frac{\partial^2 f}{\partial x^2} \cancel{=} 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y} + \left(1+\left(\frac{\partial f}{\partial x}\right)^2\right)\frac{\partial^2 f}{\partial y^2}.$$

This clearly annihilates a constant fcn. $f(x,y) = c$.