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Section 6.1

1. Let $\varepsilon > 0$. Set $\delta = \varepsilon / c$.

Then, for $\|x - y\| < \delta$,

$$\|f(\underline{\underline{x}}) - f(\underline{\underline{y}})\| \le c\,\|\underline{\underline{x}} - \underline{\underline{y}}\|$$
$$< c\,\delta = \varepsilon.$$

So f is continuous.

Fix $\underline{\underline{x}}, \underline{\underline{x}}'$ are both fixed points then

$$\|\underline{\underline{x}} - \underline{\underline{x}}'\| = \|f(\underline{\underline{x}}) - f(\underline{\underline{x}}')\|$$
$$\le c\,\|\underline{\underline{x}} - \underline{\underline{x}}'\|.$$

Since $c < 1$, this is false unless $\|\underline{\underline{x}} - \underline{\underline{x}}'\| = 0$

so $\underline{\underline{x}} = \underline{\underline{x}}'$.

2. If $\sqrt{x^2+1} = x$

then $x^2 + 1 = x^2$

so $1 = 0$. Contradiction.

While $|f'(x)| < 1$ for all x, it is also true that

$f'(x) = \dfrac{x}{\sqrt{x^2+1}}$ which is arbitrarily close to 1

as $x \to \infty$. So there will be no $c < 1$.

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6 a) Define

$$J = \sum_{k=0}^{\infty} H^k = I + H + H^2 + \cdots$$

Since $\|H\| = c < 1$, we see that

$$\|H \circ H\| \leq \|H\| \, \|H\| = c^2$$

and, in general,

$$\|H^j\| \leq c^j.$$

So $\sum_{k=0}^{\infty} H^k$ converges in norm. Moreover,

$$H \circ J = \sum_{k=0}^{\infty} H^{k+1}$$

So $(I - H) \circ J = \sum_{k=0}^{\infty} H^k - \sum_{k=0}^{\infty} H^{k+1} = I.$

9, b) $g(x) = x^3 - 2, \quad x_0 = 5/4$

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)} = \frac{5}{4} - \frac{125/64 - 2}{3 \cdot 25/16} = \frac{5}{4} - \frac{5}{12} + \frac{32}{75}$$

$$= \frac{5}{6} + \frac{32}{75} = \frac{63}{50} = 1.26$$

$$x_2 = x_1 - \frac{g(x_1)}{g'(x_1)} = \frac{63}{50} - \frac{\left(\frac{63}{50}\right)^3 - 2}{3\left(\frac{63}{50}\right)^2}$$

$$= \frac{63}{50} - \frac{2.0004 - 2}{4.76} = 1.26 - .000084 = 1.2599$$

Note that $|g'| = |3x^2| \geq 3$ on $[1, 1.5]$

and $|g''| = |6x| \leq 9$ on $[2, 1.5]$

If we work in an interval on which $|g| \leq 3^2/9 = 1$

then we will converge to a root.

Such a interval is $[1, 1.3]$.

Section 6.2

1.b) $$J_{ac} f = \begin{pmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{-2xy}{(x^2+y^2)^2} \\ \frac{-2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{pmatrix}$$

$$\det J_{ac} f = \frac{(y^2-x^2)(x^2-y^2)}{(x^2+y^2)^4} - \frac{4x^2y^2}{(x^2+y^2)^4}$$

$$= \frac{-x^4-y^4+2x^2y^2-4x^2y^2}{(x^2+y^2)^4}$$

$$= -\frac{(x^2+y^2)^2}{(x^2+y^2)^4} = -\frac{1}{(x^2+y^2)^2}$$

This is non-vanishing at every point but $(0,0)$.

So there will be a local inverse near every point

but $(0,0)$.

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d) $$\underline{\underline{\mathrm{Jac}\, f}} = \begin{pmatrix} 1 & e^y \\ e^x & 1 \end{pmatrix}$$

$$\det \underline{\underline{\mathrm{Jac}\, f}} = 1 - e^{x+y}.$$

This is nonzero, hence gives rise to a local inverse, except at point where $x = -y$.

2. a) let $\alpha = u + v$

$\beta = uv$

Then $u = \frac{\beta}{v}$ so $\alpha = \frac{\beta}{v} + v$

$$\alpha v = \beta + v^2$$

$$v^2 - \alpha v + \beta = 0$$

$$v = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}$$

Since $0 < v < u$ it is clear that $\alpha^2 - 4\beta > 0$.

For such α, β we can take

$$u = \frac{\alpha \mp \sqrt{\alpha^2 - 4\beta}}{2} \quad (*)$$

$$v = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}. \quad (*)$$

That is $\underline{\underline{g}}(\alpha, \beta) = \begin{pmatrix} \frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2} \\ \frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \end{pmatrix}$ for example.

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b) $$\underline{\underline{D}}g = \begin{pmatrix} \frac{1}{2} - \frac{\alpha}{2\sqrt{\alpha^2-4\beta}} & \frac{1}{\sqrt{\alpha^2-4\beta}} \\ \frac{1}{2} + \frac{\alpha}{2\sqrt{\alpha^2-4\beta}} & -\frac{1}{\sqrt{\alpha^2-4\beta}} \end{pmatrix} \quad (+).$$

On the other hand,

$$\underline{\underline{D}}f = \begin{pmatrix} 1 & 1 \\ v & u \end{pmatrix}$$

$$Df^{-1} = \begin{pmatrix} \frac{u}{u-v} & \frac{-1}{u-v} \\ \frac{-v}{u-v} & \frac{1}{u-v} \end{pmatrix} \quad (\star)$$

If you plug $(*)$ and $(**)$ into $(\star)$ then you will get $(+)$.

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(6)

3 b) $\frac{\partial F}{\partial y} = x_1 e^{x_1 y} + 2y \cos x_1 x_2$

At $(1, 2, 0)$ this equals $1 \cdot e^0 + 2 \cdot 0 \cdot \cos 2 = 1$.

It is nonvanishing, so can solve for y in terms of $\underline{\underline{x}}$.

Note that $\frac{\partial F}{\partial \underline{\underline{x}}} = (y e^{x_1 y} - x_2 y^2 \cos x_1 x_2, \; x_1 e^{x_1 y} - x_1 y^2 \cos x_1 x_2)$

$$D\phi(\underline{\underline{x}}) = -\left(\frac{\partial F}{\partial y}\begin{pmatrix}\underline{\underline{x}}\\ \phi(\underline{\underline{x}})\end{pmatrix}\right)^{-1} \frac{\partial F}{\partial \underline{\underline{x}}}\begin{pmatrix}\underline{\underline{x}}\\ \phi(\underline{\underline{x}})\end{pmatrix}.$$

d) $\frac{\partial \underline{\underline{F}}}{\partial \underline{\underline{y}}} = \begin{pmatrix} -2y_1 & -2y_2 \\ -1 & 1 \end{pmatrix}$.

At $(2, 1, 1)$, this equals $\begin{pmatrix} -2 & -2 \\ -1 & 1 \end{pmatrix}$.

This matrix has nonzero determinant so is certainly invertible. Hence can solve for $\underline{\underline{y}}$ in terms of x.

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$$\frac{\partial F}{\partial x} = \begin{pmatrix} 2x & 1 \end{pmatrix}.$$

$$D\phi(x) = -\left(\frac{\partial \underline{F}}{\partial \underline{y}}\begin{pmatrix} x \\ \phi(\underline{x}) \end{pmatrix}\right)^{-1} \frac{\partial \underline{F}}{\partial x}\begin{pmatrix} x \\ \phi(x) \end{pmatrix}.$$