SOLUTIONS

Math 318 Krantz
February 20, 2013

First Midterm

General Instructions: Read each problem carefully. Do only what is requested—nothing more nor less. *Always show your work on each problem*—unless the problem explicitly tells you not to show work. You will not get full credit for just writing down the answer. The point value of each problem is shown. The total points on this exam is 100. Ask questions if anything is unclear.

1. (10 points) Give an explicit verification that

$$V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 3x_1 - 2x_2 + x_3 = 0\}$$

is a subspace of $\mathbb{R}^3$.

a) $3 \cdot 0 - 2 \cdot 0 + 0 = 0$ so $0 \in V$

b) If $(x_1, x_2, x_3), (x_1', x_2', x_3') \in V$, then

$$3x_1 - 2x_2 + x_3 = 0$$
$$3x_1' - 2x_2' + x_3' = 0$$

adding: $6(x_1 + x_1') - 2(x_2 + x_2') + (x_3 + x_3') = 0$

So $(x_1, x_2, x_3) + (x_1', x_2', x_3') \in V$.

c) If $(x_1, x_2, x_3) \in V$ and $c \in \mathbb{R}$, then

$$3x_1 - 2x_2 + x_3 = 0$$

hence $3(cx_1) - 2(cx_2) + cx_3 = 0$.

Thus $c(x_1, x_2, x_3) \in V$.

2. (10 points) Give the formal definition of a linear transformation.

A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear if

a) $T(\underline{v} + \underline{w}) = T(\underline{v}) + T(\underline{w}) \quad \forall \underline{v}, \underline{w} \in \mathbb{R}^n$;

b) $T(c\underline{v}) = c\,T(\underline{v}) \quad \forall \underline{v} \in \mathbb{R}^n \ \forall c \in \mathbb{R}$

3. (10 points) Find the equation of the plane that passes through $(2, 5, 3)$ and is perpendicular to the vector $\mathbf{v} = \langle 1, 1, 2 \rangle$.

Let $\underline{X} = (x_1, x_2, x_3)$ be any point on the plane. Let $P = (2, 5, 3)$. Then the vector $\overrightarrow{P\underline{X}}$ is $\perp$ to $\langle 1, 1, 2 \rangle$. So

$$\overrightarrow{P\underline{X}} \cdot \langle 1, 1, 2 \rangle = 0$$

$$\langle x_1 - 2, x_2 - 5, x_3 - 3 \rangle \cdot \langle 1, 1, 2 \rangle = 0$$

$$(x_1 - 2) + (x_2 - 5) + 2(x_3 - 3) = 0$$

$$x_1 + x_2 + 2x_3 = 13$$

4. (10 points) Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix}.$$

Calculate AB and BA.

$$AB = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 4 & -2 \\ -2 & 1 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 2 & 2 & -2 \\ 1 & 0 & 4 \end{bmatrix}$$

5. (10 points) Find a nonzero vector in $\mathbb{R}^3$ which is perpendicular to both $\mathbf{u} = \langle 2, 4, 1 \rangle$ and $\mathbf{v} = \langle -1, 0, 1 \rangle$.

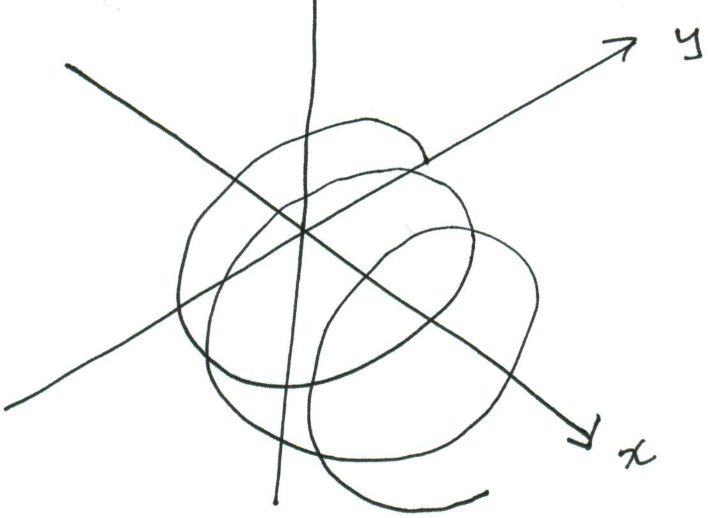
$$\underline{u} \times \underline{v} = \det \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \underline{i}\, 4 - \underline{j}\, 3 + \underline{k}\, 4$$

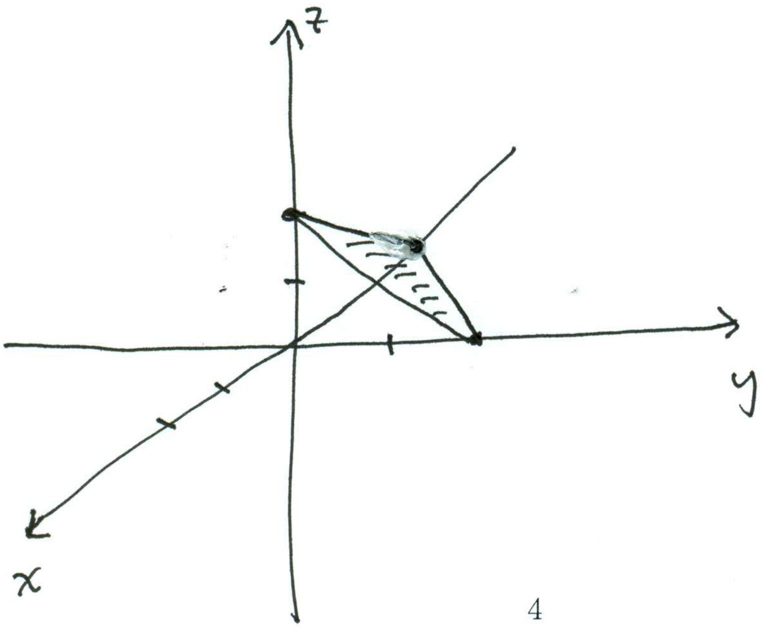
6. (10 points) Sketch the trace of the parametrized curve

$$\mathbf{r}(t) = \langle t, \cos t, \sin t \rangle$$

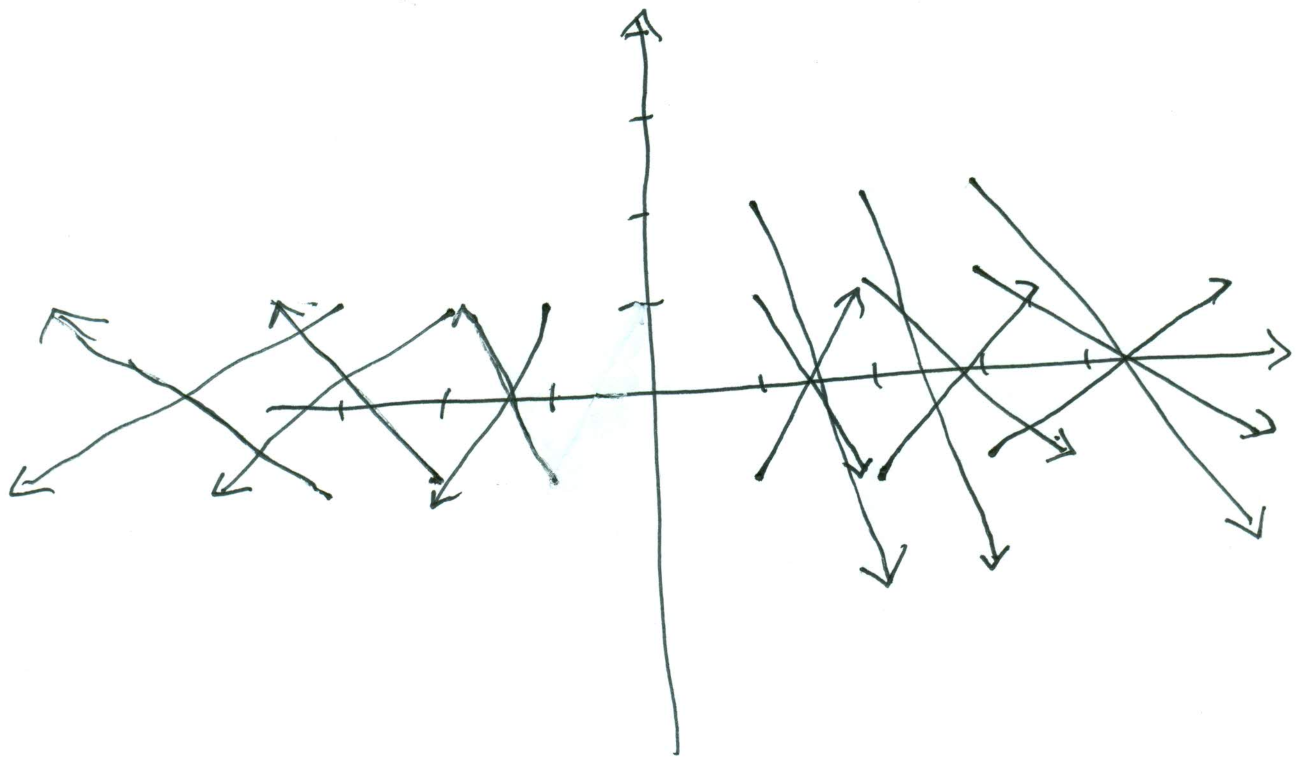
in $\mathbb{R}^3$.



7. (8 points) Sketch the graph (which is a 2-dimensional surface in $\mathbb{R}^3$) of the function $f(x, y) = x - y + 2$.



8. (10 points) Sketch the vector field in the plane given by $\mathbf{f}(x, y) = \langle x, -2y \rangle$. [**Hint:** Your answer should be a set of axes with a bunch of arrows exhibiting the flow of the vector field.]



9. Which of the following sets is open and which is closed? Which is neither? Give brief reasons for your answers.

(a) (3 points) $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$

Open because no boundary. If $(x,y) \in S$ and ε is distance to circle, then $B((x,y), \varepsilon/2)$ lies in S.

(b) (3 points) $T = \{(x, y) \in \mathbb{R}^2 : 1 \leq x < 2\}$. Neither open nor closed.

Not open because $(1, 0)$ has no neighborhood that lies in T. Not closed because $z_j = 2 - 1/j$ has limit point which is not in T.

(c) (3 points) $U = \{(x, y, z) \in \mathbb{R}^3 : -1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1\}$

Closed because the complement is open.

(d) (3 points) $V = \{(x, y, z) \in \mathbb{R}^3 : 1 \leq x^2 + y^2 + z^2 < 2\}$ Neither open nor closed. Not open because the point $(1, 0, 0)$ has no neighborhood that lies in V. Not closed because $z_j = (\sqrt{2} - 1/j, 0, 0)$ has limit point not in V.

10. (10 points) Explain why this function f has no limit at the origin:

$$f(x,y) = \frac{xy}{x^2+y^2}\,.$$

$$\lim_{t \to 0} f(t,0) = \lim \frac{0}{t^2+0^2} = 0$$

$$\lim_{t \to 0} f(t,t) = \lim \frac{t \cdot t}{t^2+t^2} = \frac{1}{2}$$

These are unequal.