MATH 4111

Solutions to Practice Final Exam

1. Let $z = x+iy$, $w = u+iv$. Then

$$z \cdot w = (xu - yv) + i(xv + yu)$$

$$w \cdot z = (ux - vy) + i(uy + vx).$$

These are equal.

2. Let S be bounded above and let σ be a supremum of S. If τ is another supremum and $\tau \neq \sigma$, say that $\tau < \sigma$. This would mean that σ is <u>not</u> <u>the least</u> upper bound. Contradiction. Same contradiction if $\sigma < \tau$. So $\sigma = \tau$.

Same reasoning for the infimum.

3. $\dfrac{\cos^2 j}{j} = \dfrac{1 + \cos 2j}{j}$. The series $\sum_j \dfrac{\cos 2j}{j}$ converges by Abel's Test as in Example 3.35. And the series $\sum_j \dfrac{1}{j}$ diverges. So $\sum_j \dfrac{\cos^2 j}{j}$ diverges.

4. For $j \geq 9$,

$$\frac{j!}{2^j} \geq \frac{j(j-1)\cdots 9}{2^j} \geq \frac{(2^3)^{j-8}}{2^j} = 2^{2j-24} \to \infty$$

as $j \to \infty$.

5. By the integral test,

$$\int_1^\infty \frac{1}{x\log x}\,dx = \log\log x \Big]_1^\infty = \infty$$

so integral diverges and series diverges.

6. The series $\sum_{j=1}^{\infty} \frac{1}{j^{3/4}}$ diverges but $\sum_j \frac{1}{j^{3/4}} \cdot \frac{1}{j^{1/2}} = \sum_j \frac{1}{j^{5/4}}$ converges.

The series $\sum_{j=1}^{\infty} \frac{1}{j^{1/4}}$ diverges and $\sum_j \frac{1}{j^{1/4}} \cdot \frac{1}{j^{1/2}} = \sum_j \frac{1}{j^{3/4}}$ also diverges.

7.
$$f''(x) \le x$$

$$\int_0^t f''(x)\,dx \le \int_0^t x\,dx$$

$$f'(t) - f(0) \le \frac{t^2}{2}$$

$$\int_0^w f'(t) - f(0)\,dt \le \int_0^w \frac{t^2}{2}\,dt$$

$$f(w) - f(0)w - f'(0) \le \frac{w^3}{6}$$

$$f(w) \le f(0)w + f'(0) + \frac{w^3}{6}$$

So f grows at most cubically at ∞.

8. Let $f(x) = \sqrt{x}$. So we are looking at

$$f(x+1) - f(x) = ((x+1) - x) \cdot f'(\xi) \qquad (*)$$

for some ξ between x and $x+1$.

But $f'(x) = \frac{1}{2} x^{-1/2}$ so

$$|f'(\xi)| \le \frac{1}{2}|x^{-1/2}| .$$

Hence

$$|f(x+1) - f(x)| \le 1 \cdot \frac{1}{2}|x|^{-1/2} \to 0$$

as $x \to +\infty$.

9. Let $O_j = (-1/2j, 1 + 1/2j)$, $j = 2, 3, \ldots$.

Then $\bigcap_j O_j = [0, 1]$ which is closed.

Let $U_j = (-1/2j, 1)$, $j = 2, 3, \ldots$.

Then $\bigcap_j U_j = [0, 1)$ which is half-open.

Let $V_j = (\frac{1}{j}, \infty)$, $j = 2, 3, \ldots$

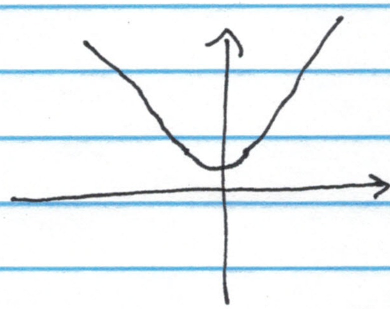
Then $\bigcap_j V_j = (1, \infty)$ which is open.

10. Let $p_j = \frac{(-1)^j x^{2j+1}}{(2j+1)!}$. These are polynomials.

But $\sum_{j=0}^{\infty} p_j = \sin x$, which is not a polynomial.

11. Let

$$f_j(x) = \begin{cases} \frac{1}{\sqrt{j}} + \frac{1}{j} + 2\left(x - \frac{1}{\sqrt{j}}\right) & \text{if } x \geq \frac{1}{\sqrt{j}} \\ \sqrt{j}\, x^2 + \frac{1}{j} & \text{if } -\frac{1}{\sqrt{j}} < x < \frac{1}{\sqrt{j}} \\ \frac{1}{\sqrt{j}} + \frac{1}{j} - 2\left(x + \frac{1}{\sqrt{j}}\right) & \text{if } x \leq -\frac{1}{\sqrt{j}}, \end{cases}$$



Then f_j is obviously continuous and

$$f_j \to f = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$$

uniformly on $[-1, 1]$.

Also

$$f_j'(x) = \begin{cases} 2 & \text{if } x \geq \frac{1}{\sqrt{j}} \\ 2\sqrt{j}\, x & \text{if } -\frac{1}{\sqrt{j}} < x < \frac{1}{\sqrt{j}} \\ -2 & \text{if } x \leq -\frac{1}{\sqrt{j}} \end{cases}$$

is continuous on $[-1, 1]$. But f is not differentiable at 0.

12. We may as well use a partition with mesh less than 1. The only intervals that are non-zero in the calculation of the

Riemann-Stieltjes sum are those that contain 1, 2, 3, 4, 5. And then $\Delta\alpha_j = 1$. So the value of the integral is

$$\cos 1 + \cos 2 + \cos 3 + \cos 4 + \cos 5.$$

13. $$\int_0^1 f(x)\,dg(x) = \int_0^1 f(x) g'(x)\,dx$$
$$= \int_0^1 x^2 \cos x\,dx$$
$$= x^2 \sin x \Big]_0^1 - \int_0^1 2x \sin x\,dx$$
$$= \sin 1 - \left[-2x\cos x \Big]_0^1 + \int_0^1 2\cos x\,dx \right]$$
$$= \sin 1 + 2\cos 1 - 0 + 2\sin x \Big]_0^1$$
$$= 3\sin 1 + 2\cos 1.$$

14. $$\left| \sum_{j=M}^{N} \frac{\cos jx}{j^3} \right| \le \sum_{j=M}^{N} \frac{1}{j^3} < \varepsilon \quad \text{for } M, N$$

large enough by the integral test. So the series is uniformly Cauchy. Hence the series converges uniformly. Therefore the sum is a continuous function.