Math 4111 Sols. to First HW Asst.

§1.1

1. The set $(0, 1]$ contains its lub 1 but not its glb 0.

   The set $[0, 1)$ contains its glb 0 but not its lub 1.

2. The set $(0, \infty)$ does not have a lub. The set $(-\infty, 0)$ does not have a glb.

3. $A \subset \mathbb{R}$ is bounded above, $\alpha = \sup A$. So $\alpha \geq a$ $\forall a \in A$ and if $b < \alpha$ then b is <u>not</u> an upper bound for A. Thus $-\alpha \leq -a$ $\forall a \in A$ and if $-\alpha < c$ then c is not a lower bound for $-A$. So $-\alpha$ is the glb of $-A$.

   If $B \subset \mathbb{R}$ is bounded below and $\beta = \inf B$, then $\beta \leq b$ $\forall b \in B$. Also if $c > \beta$ then c is <u>not</u> a lower bound for B. Thus $-\beta \geq -b$ $\forall b \in B$ and if $d < -\beta$ then d is not an upper bound for $-B$.

4. The lub for this set is $\sqrt{2}$ which is a real number but not a rational number.

5. If a_1, a_2 are both l.u.b. for A and are distinct then one is less than the other. So, for instance, $a_1 < a_2$. But then a_1 cannot be an upper bound for A. Contradiction.

Similar for glb.

7. We want to prove $|x+y| \leq |x| + |y|$.

Case 1: $x \geq 0, y \geq 0$:

$|x+y| = x+y = |x| + |y|$. ✓

Case 2: $x \geq 0, y < 0$:

$x + y \leq x + (-y) = |x| + |-y| = |x| + |y|$

$-(x+y) \geq -x + y = -|x| - |y| = -(|x|+|y|)$.

So $|x+y| \leq |x| + |y|$

Case 3: $x < 0, y \geq 0$:

Similar to Case 2.

Case 4: $x < 0, y < 0$.

$x + y \leq -x + (-y) = |x| + |y|$

$x + y = -|x| + -|y| = -(|x| + |y|)$

So $|x+y| \leq |x| + |y|$

Altogether then, $|x+y| \leq |x|+|y|$.

11. $\left\{\frac{\sqrt{2}}{2^j} : j = 1, 2, \ldots\right\}$.

§1.2

2. $\overline{\left(\frac{z}{w}\right)} = \overline{\left(z \cdot \frac{\overline{w}}{|w|^2}\right)} = \overline{z} \cdot \frac{w}{|w|^2} = \overline{z} \cdot \frac{1}{\overline{w}} = \frac{\overline{z}}{\overline{w}}$.

3. $(re^{i\theta})^3 = 1+i = \sqrt{2}e^{i\pi/4}$

$r^3e^{i3\theta} = \sqrt{2}e^{i\pi/4}$

$r = 2^{1/6}, \theta = \frac{i\pi}{12} \qquad z_1 = 2^{1/6}e^{i\pi/12}$

$(re^{i\theta})^3 = \sqrt{2}e^{i(\frac{\pi}{4}+2\pi)} = \sqrt{2}e^{i(\frac{9\pi}{4})}$

$r^3e^{i3\theta} = \sqrt{2}e^{i(\frac{9\pi}{4})}$

$r = 2^{1/6}, \theta = \frac{3\pi}{4} \qquad z_2 = 2^{1/6}e^{i\frac{3\pi}{4}}$

$(re^{i\theta})^3 = \sqrt{2}e^{i(\frac{\pi}{4}+4\pi)} = \sqrt{2}e^{i(\frac{17\pi}{4})}$

$r^3e^{3i\theta} = \sqrt{2}e^{i\frac{17\pi}{4}}$

$r = 2^{1/6}, \theta = \frac{17\pi}{12} \qquad z_3 = 2^{1/6}e^{i\frac{17\pi}{12}}$

5. $\phi : \mathbb{R} \to \mathbb{C}$

$\phi(x) = x+i0$

$\phi(x+y) = (x+y)+i0 = (x+i0)+(y+i0) = \phi(x)+\phi(y)$

$\phi(x\cdot y) = x\cdot y+i0 = (x+i0)\cdot(y+i0) = \phi(x)\cdot\phi(y)$

4. $(x+iy)+(x'+iy') = (x+x')+i(y+y')$

$$= (x'+x)+i(y'+y)$$

$$= (x'+iy')+(x+iy)$$

$$(x+iy)\cdot(x'+iy') = (xx'-yy')+i(xy'+yx')$$

$$= (x'x-y'y)+i(x'y+y'x)$$

$$= (x'+iy')\cdot(x+iy)$$

$$(x+iy)((a+ib)+(c+id)) = (x+iy)\cdot((a+c)+i(b+d))$$

$$= [x(a+c)-y(b+d)]$$

$$+i[y(a+c)+x(b+d)] \quad (*)$$

$$(x+iy)(a+ib)+(x+iy)(c+id) =$$

$$xa-yb+ixb+iya+xc-yd+iyc+ixd$$

$$= [x(a+c)-y(b+d)]$$

$$+i[y(a+c)+x(b+d)] \quad (**)$$

Note That $(*) = (**)$.

7. Let z_0 be a root of the polynomial

$$a_0+a_1z+a_2z^2+\cdots+a_kz^k$$

with real coefficients. So each a_j is real and

$$a_0+a_1z_0+a_2z_0^2+\cdots+a_kz_0^k = 0$$

Hence $\overline{a_0}+\overline{a_1}\,\overline{z_0}+\overline{a_2}\,\overline{z_0}^2+\cdots+\overline{a_k}\,\overline{z_0}^k = 0.$

But each $\overline{a_j} = a_j$. So

$$a_0 + a_1\overline{z_0} + a_2\overline{z_0}^2 + \cdots + a_k\overline{z_0}^k = 0.$$

Hence $\overline{z_0}$ is a root of the polynomial.

8. $i = e^{i\frac{\pi}{2}}$

$$(re^{i\theta})^2 = 1 \cdot e^{i\frac{\pi}{2}}$$

$$r^2e^{i2\theta} = 1 \cdot e^{i\frac{\pi}{2}}$$

$$r = 1,\ \theta = \frac{\pi}{4} \qquad z_1 = 1 \cdot e^{i\frac{\pi}{4}}$$

$$(re^{i\theta})^2 = e^{i\left(\frac{\pi}{2}+2\pi\right)} = e^{i\left(\frac{5\pi}{2}\right)}$$

$$r^2e^{i2\theta} = 1 \cdot e^{i\left(\frac{5\pi}{2}\right)}$$

$$r = 1,\ \theta = \frac{5\pi}{4}. \qquad z_2 = 1 \cdot e^{i\frac{5\pi}{4}}.$$

9. $\mathbb{C} \supset \mathbb{R}$ and $\mathbb{R}$ is uncountable so $\mathbb{C}$ is uncountable.

10. Let α be a nonzero complex number. The solutions of the polynomial

$$z^k - \alpha = 0$$

are the k^{th} roots of α. By the fundamental theorem of algebra, there are k of them.

13, This is the same as #3.