MATH 4111 SOLUTIONS TO HW 2

§2.1 1. Let $a_j = n$ for $2^n \leq j \leq 2^{n+1}$.

Then there are strings of length 2, 4, 8, etc. but the sequence diverges to $+\infty$.

2. If the β_j are bounded then there is a subsequence β_{j_k} that are constantly equal to some integer n. So now we are looking at

$$a_{j_k} = \frac{\alpha_{j_k}}{n}.$$

If infinitely many of the α_{j_k} are the same, then that constant subsequence converges to a rational number. Otherwise a_{j_k} does not converge.

In any event, we see that if $\{\beta_j\}$ is bounded then the sequence cannot converge to an irrational number.

3. The rational numbers with denominator a power of 2 are dense in the real line. So any real number is the limit of such a sequence.

8. (2) If $a_j \to \alpha$, $b_j \to \beta$ then $a_j + b_j \to \alpha + \beta$.

<u>Proof</u>: Let $\varepsilon > 0$. Choose N_1 so large that $j > N_1 \Rightarrow |a_j - \alpha| < \frac{\varepsilon}{2}$. Choose N_2 so large that $j > N_2 \Rightarrow |b_j - \beta| < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. If $j > N$ then

$$|(a_j + b_j) - (\alpha + \beta)| \le |a_j - \alpha| + |b_j - \beta|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $a_j + b_j \to \alpha + \beta$.

(4) If $a_j \to \alpha$, $b_j \to \beta$, $b_j \neq 0$, $\beta \neq 0$, then
$$\frac{a_j}{b_j} \to \frac{\alpha}{\beta}.$$

<u>Proof</u>: Let $\varepsilon > 0$. Choose N_1 so large that $j > N_1 \Rightarrow |b_j - \beta| < \frac{|\beta|}{2}$. Choose N_2 so large that $j > N_2 \Rightarrow |a_j - \alpha| < \frac{\varepsilon}{2|\beta|}$. Choose N_3 so large that $j > N_3 \Rightarrow |b_j - \beta| < \frac{\varepsilon}{2|\alpha|}$. Let $j > \max\{N_1, N_2, N_3\}$. Then

$$\left|\frac{a_j}{b_j} - \frac{\alpha}{\beta}\right| = \left|\frac{a_j\beta - \alpha b_j}{\beta b_j}\right| = \left|\frac{a_j\beta - \alpha\beta + \alpha\beta - \alpha b_j}{\beta \cdot b_j}\right|$$
$$= \left|\frac{\beta(a_j - \alpha)}{\beta b_j} + \frac{\alpha(\beta - b_j)}{\beta b_j}\right| \le \frac{|a_j - \alpha|}{|b_j|} + \left|\frac{\alpha}{\beta}\right| \frac{|\beta - b_j|}{|b_j|}$$
$$\le \frac{\varepsilon / 2|\beta|}{|\beta|/2} + \frac{|\alpha|}{|\beta|} \cdot \frac{\varepsilon / 2|\alpha|}{|\beta|/2}$$
$$= \varepsilon\left(\frac{1}{2|\beta|^2} + \frac{1}{2|\beta|^2}\right) = \frac{\varepsilon}{|\beta|^2}. \text{ So } \frac{a_j}{b_j} \to \frac{\alpha}{\beta}.$$

10. Suppose not. Then $\exists\ \varepsilon > 0$ such that $s \leq t - \varepsilon$ for all $s \in S$. But then $\sup S \leq t - \varepsilon$, which is a contradiction.

11. Suppose not. Then there exist $\varepsilon > 0$ and $a_{j_1} < a_{j_2} < a_{j_3} < \cdots$ such that $|a_{j_k} - \alpha| > \varepsilon\ \forall k$. But then $\{a_{j_k}\}$ does not have a subsequence that converges to α.

12. If the sequence is not Cauchy, then $\exists\ \varepsilon > 0$ s.t. $|a_m - a_n| > \varepsilon$ for infinitely many large m, n. Let $N > \frac{1}{\varepsilon}$. If we choose N such pairs m_j, n_j, then

$$\sum |a_{m_j} - a_{n_j}| \geq N \cdot \varepsilon > \frac{1}{\varepsilon} \cdot \varepsilon = 1.$$

Contradiction.

Section 2.2

1. Let a_j be a decreasing sequence that is bounded below. So $a_1 \geq a_2 \geq a_3 \geq \dots \alpha$. Thus
$$\alpha \leq a_j \leq a_1 \quad \forall j.$$
So $\{a_j\}$ is a bounded sequence. By Bolzano Weierstrass, $\exists$ a subsequence a_{j_k} which converges to some finite β.

Let $\epsilon > 0$. Choose N so large that $k > N \Rightarrow |a_{j_k} - \beta| < \epsilon$. But then, if $j > a_{j_k}$, $|a_j - \beta| < \epsilon$. So the full sequence converges to β.

2. Let $\{q_j\}$ be an enumeration of the rationals. Consider the sequence

(*) $q_1, q_1, q_2, q_1, q_2, q_3, q_1, q_2, q_3, q_4, \dots$

If α is any real number then there is certainly a sequence r_j of rationals with $r_j \to \alpha$. But the sequence $\{r_j\}$ is a subsequence of (*). Same if $\alpha = \pm\infty$.

4. $x_{j+1} = x_j - \frac{x_j^2 - 2}{2x_j}$

Notice That

$$(x_{j+1})^2 = x_j^2 - (x_j^2 - 2) + (x_j^2-2)^2$$
$$= 2 + (x_j^2 - 2)^2 \geq 2 \quad \forall j. \qquad (*)$$

So $x_{j+1} = x_j - \frac{x_j^2 - 2}{2x_j} \leq x_j$. So sequence is decreasing

(*) Shows that the sequence is bounded below.

So the sequence converges to some α.

We can write

$$x_{j+1} = \frac{x_j}{2} + \frac{1}{x_j}.$$

Letting $j \to \infty$ gives

$$\alpha = \frac{\alpha}{2} + \frac{1}{\alpha}$$

$$\alpha^2 = \frac{\alpha^2}{2} + 1$$

$$\frac{\alpha^2}{2} = 1$$

$$\alpha^2 = 2$$

$\alpha = \sqrt{2}$ is the limit.

8. There are infinitely many elements $n \bmod \pi$, all contained in the interval $[0, \pi]$. By Bolzano-Weierstrass, there is a subsequence $n_j \bmod \pi$ that converges to some $n_0 \bmod \pi$.

But then $(n_j - n_0) \bmod \pi \to 0 \bmod \pi$.
So the elements $(n_j - n_0) \bmod \pi$ are arbitrarily small. But then the elements $k(n-n_0) \bmod \pi$, $k \in \mathbb{N}$ are dense in $[0, \pi]$.