Math 4111 SOLUTIONS TO HW4

§4.2

1. Let S be a set and $T = \{t \in \mathbb{R} : |s-t| < \varepsilon$ for some $s \in S\}$.

We claim that T is open. Let $t \in T$. Then there is an $s \in S$ such that $|s-t| = \eta < \varepsilon$. Let u be a point such that $|u-t| < \varepsilon - \eta$. Then

$$|u-s| \le |u-t| + |t-s| < (\varepsilon - \eta) + \eta = \varepsilon.$$

So $u \in T$. Hence $(t-(\varepsilon-\eta), t+(\varepsilon-\eta)) \subseteq T$. So T is open.

3. The set $[0,1)$ is neither open nor closed.

5. Let $X_j = [j, \infty)$. Then

$$X_1 \supseteq X_2 \supseteq \cdots, \text{ yet } \bigcap_{j=1}^{\infty} X_j = \emptyset.$$

And each X_j is closed.

7. Let $U_j = (-\frac{1}{j}, 1+\frac{1}{j})$. Then each U_j is open and

$$\bigcap_{j=1}^{\infty} U_j = [0,1], \text{ which is closed.}$$

§4.2

1. Let S be any set of real numbers. The closure of S is $\overline{S} = S \cup \partial S$. So obviously $S \subseteq \overline{S}$.

Let $x \in$ complement of $\overline{S}$. So $x \notin S$ and $x \notin \partial S$. Thus there is a neighborhood of x that does not intersect both S and ${}^cS$. That neighborhood obviously cannot intersect ∂S. And it does not intersect S. So it lies in ${}^c(S \cup \partial S)$. Thus ${}^c(S \cup \partial S)$ is open. Hence $\overline{S} = S \cup \partial S$ is closed.

Let $y \in \partial S$. Then of course $y \in \overline{S}$ and y cannot be in the interior of S because no neighborhood of y lies entirely in S. So $\partial S \subseteq \overline{S} \setminus \mathring{S}$.

Now let $z \in \overline{S} \setminus \mathring{S}$. So no neighborhood of z lies entirely in S. So every neighborhood of z intersects ${}^cS$. Since $\overline{S} = S \cup \partial S$, z either lies in S or in ∂S. In the first case, every neighborhood of z contains a point of S (namely z). In the second case, every neighborhood of z contains a point of S (that is the def'n of boundary point). So $z \in \partial S$. Thus $\overline{S} \setminus \mathring{S} \subseteq \partial S$.

2. $S = \{1, 1/2, 1/3, \ldots\} \cup \{0\}$.

$\mathring{S} = \emptyset$

$\partial S = S$.

3. Let $E_j = [1/j, 1 - 1/j]$. Then each E_j is closed and $\bigcup_{j=1}^{\infty} E_j = (0, 1)$, which is open.

Instead let $F_j = [0, 1]$. Then each F_j is closed and $\bigcup_{j=1}^{\infty} F_j = [0, 1]$, which is closed.

5. Let $S \subseteq \mathbb{R}$. Let $s \in \mathring{S}$. So $\exists \varepsilon > 0$ such that $(s - \varepsilon, s + \varepsilon) \subseteq S$. Let $t \in (s - \varepsilon, s + \varepsilon)$. So $|t - s| = \eta < \varepsilon$. If $|u - t| < \varepsilon - \eta$, then $|u - s| \le |u - t| + |t - s| < (\varepsilon - \eta) + \eta = \varepsilon$ so $u \in \mathring{S}$. Hence $\mathring{S}$ is open.

Let S be open. Let $s \in S$. Then $\exists \varepsilon > 0$ so that $(s - \varepsilon, s + \varepsilon) \subseteq S$. So $s \in \mathring{S}$. Hence $S \subseteq \mathring{S}$. Now let $t \in \mathring{S}$. Then $\exists \delta > 0$ such that $(t - \delta, t + \delta) \subseteq S$. Hence $t \in S$. Thus $\mathring{S} \subseteq S$. In conclusion, $S = \mathring{S}$.

Conversely, assume $S = \mathring{S}$. Let $x \in S$. Then $x \in \mathring{S}$. So $\exists \varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq S$. So S is open.

§4.3

1. Let K be compact and E closed. Then K is closed and bounded. So $K \cap E$ is closed. Also $K \cap E \subseteq K$, so $K \cap E$ is bounded. Hence $K \cap E$ is compact.

2. Let K be compact and U open such that $U \supseteq K$. If $k \in K$ then $k \in U$ so $\exists \varepsilon > 0$ such that $(k-\varepsilon_k, k+\varepsilon_k) = I_k \subseteq U$. The Let $J_k = (k-\varepsilon_k/2, k+\varepsilon_k/2)$. Then the intervals J_k form an open cover of K. So there is a finite subcover

$$(k_1-\varepsilon_{k_1}/2, k_1+\varepsilon_{k_1}/2), (k_2-\varepsilon_{k_2}/2, k_2+\varepsilon_{k_2}/2), \ldots, (k_m-\varepsilon_{k_m}/2, k_m+\varepsilon_{k_m}/2).$$

Let $\varepsilon = \min\{\varepsilon_{k_1}/2, \varepsilon_{k_2}/2, \ldots, \varepsilon_{k_m}/2\}$.

If $x \in K$ is any point then $x \in J_{k_\ell}$ for some $\ell = 1, \ldots, m$. So if $|t-x| < \varepsilon$, then

$$|t-k_\ell| \le |t-x| + |x-k_\ell| < \varepsilon + \frac{\varepsilon_{k_\ell}}{2} \le \frac{\varepsilon_{k_\ell}}{2} + \frac{\varepsilon_{k_\ell}}{2} = \varepsilon_{k_\ell}.$$

Hence $t \in U$.

4. Let K be a compact set. For each $k \in K$, let $I_k = (k-\delta, k+\delta)$. Then the I_k form an open cover of K. So there is a finite subcover $I_{k_1}, I_{k_2}, \ldots, I_{k_\ell}$ of K.

That is what we seek.

8. If K is compact then K is closed and bounded. So ${}^cK$ is open and unbounded. So ${}^cK$ is not compact.

§ 4.4

1. We remove one set of length $\frac{1}{5}$

two sets of length $\frac{1}{25}$

four sets of length $\frac{1}{125}$

etc.

So the total length of all intervals removed is

$$\sum_{j=1}^{\infty} \frac{2^{j-1}}{5^j} = \frac{1}{5} \sum_{j=1}^{\infty} \left(\frac{2}{5}\right)^{j-1} = \frac{1}{5} \sum_{j=0}^{\infty} \left(\frac{2}{5}\right)^j$$

$$= \frac{1}{5} \cdot \frac{1}{1-\frac{2}{5}} = \frac{1}{5} \cdot \frac{1}{\frac{3}{5}} = \frac{1}{3}.$$

Thus the constructed Cantor-like set has length $\frac{2}{3}$.

We can assign an address of 0s and 1s to each element of this Cantor-like set, just as we did in the text. So this new Cantor set is uncountable.

This set is definitely different from the Cantor ternary set. After all, it has a different length.

3. Let $U = (0, 1)$ and $V = (1, 2)$. These are disjoint open sets. But if $\varepsilon > 0$, then $1 - \frac{\varepsilon}{3} \in U$ and $1 + \varepsilon/3 \in V$ and

$$\left|\left(1 - \frac{\varepsilon}{3}\right) - \left(1 + \frac{\varepsilon}{3}\right)\right| = \frac{2\varepsilon}{3} < \varepsilon. \text{ So}$$

$$\operatorname{dist}(U, V) < \varepsilon \text{ for every } \varepsilon > 0.$$

In conclusion, $\operatorname{dist}(U, V) = 0$.

4. Let $0 < \lambda < 1$. Set $\varepsilon = \frac{\lambda}{1 + 2\lambda}$. Construct a Cantor-like set by removing one interval of length ε at stage 1, two intervals of length ε^2 at stage 2, four intervals of length ε^3 at stage 3, etc.

So the total length of intervals removed is

$$\sum_{j=1}^{\infty} 2^{j-1}\varepsilon^{j} = \varepsilon \sum_{j=1}^{\infty} (2\varepsilon)^{j-1}$$

$$= \varepsilon \sum_{j=0}^{\infty} (2\varepsilon)^{j} = \frac{\varepsilon}{1-2\varepsilon}$$

$$= \frac{\frac{\lambda}{1+2\lambda}}{1-2\left(\frac{\lambda}{1+2\lambda}\right)} = \frac{\lambda}{1+2\lambda-2\lambda} = \lambda.$$

Thus the complement of the Cantor-like set we are constructing has length λ.

6. The Cantor set has length 0. So if $c \in C$ and $\varepsilon > 0$ then $(c-\varepsilon, c+\varepsilon)$ will intersect the complement. Hence $(c-\varepsilon, c+\varepsilon) \not\subseteq C$. Thus $\mathring{C} = \emptyset$. By the same token, if $x \in C$ then, for every $\varepsilon > 0$, $(x-\varepsilon, x+\varepsilon)$ intersects both C and ${}^cC$. So $x \in \partial C$. Since C is closed, it contains all its boundary points. So $\partial C \equiv C$.

§4.5

1. Let $F_j = {}^c U_j$. Then each F_j is closed and bounded, hence compact. Also

$$F_1 \supseteq F_2 \supseteq \cdots .$$

So $\bigcap_{j=1}^{\infty} F_j$ is nonempty. Therefore $\bigcup_{j=1}^{\infty} U_j$ cannot be all of $\mathbb{R}$.

3. Let E and F be perfect. Then each of E, F is closed and every point of each set is an accumulation point. It follows that $E \times F$ is closed and each point is an accumulation point. In particular, if $(e, f) \in E \times F$, then $\exists\, e_j \in E$ s.t. $e_j \to e$ and $\exists\, f_j \in F$ s.t. $f_j \to f$. So $(e_j, f_j) \to (e, f)$.

6. A connected set is an interval. All intervals $[a, b]$ are perfect. The others are not. So $[a, b)$, $(a, b]$ and (a, b) are connected and not perfect. The set (a, b) is imperfect because its complement $(-\infty, a] \cup [b, \infty)$ is perfect.

It is rather difficult to describe all imperfect sets.

7. The interior of a perfect set will still have every point an accumulation point. But it will not be closed.