MATH 4111

SOLUTIONS TO HW5

Section 5.1

1. Let $\lim_{x \to c} f(x) = l$, $\lim_{x \to c} g(x) = m$. Let $\varepsilon > 0$.

Choose $\delta_1 > 0$ such that $|x-c| < \delta_1 \Rightarrow |f(x) - l| < \varepsilon$.
Choose $\delta_2 > 0$ such that $|x-c| < \delta_2 \Rightarrow |g(x) - m| < \varepsilon$
Let $\delta = \min(\delta_1, \delta_2)$. Let $|x-c| < \delta$. Then

$$m = (m - g(x)) + (g(x) - f(x)) + f(x)$$
$$\geq m - g(x) + f(x)$$
$$\geq f(x) - \varepsilon.$$

So $f(x) \leq m + \varepsilon$ $\forall \varepsilon > 0$, In conclusion
$f(x) \leq m$ $\forall |x-c| < \delta$. $(*)$

Also, for such x,

$$l = (l - f(x)) + f(x)$$
$$\leq f(x) + \varepsilon.$$

Since this is true $\forall \varepsilon > 0$, we have $l \leq f(x)$ $(**)$
for $|x-c| < \delta$.

Putting together $(*)$ and $(**)$ we get $l \leq m$.

2. An isolated point has no neighboring points, so $\lim_{x \to c} f(x)$ makes no sence at such a point.

6. Let $\varepsilon > 0$. Choose $\delta > 0$ such that $|x - P| < \delta$ implies $|f(x) - l| < \varepsilon$, where l is the limit of f at P.

Now if $|h| < \delta$, then $|(P+h) - P| < \delta$ so $|f(P+h) - \ell| < \varepsilon$. Hence $\lim_{h \to 0} f(P+h) = \ell$.

The reverse direction is similar.

9. Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. If $|x - 0| < \delta$ then

$$|x \sin(1/x) - 0| = |x \sin 1/x| \le |x| < \delta = \varepsilon.$$

So $\lim_{x \to 0} x \sin(1/x) = 0$.

$$g\left(\frac{1}{n\pi}\right) = \sin n\pi = 0$$

$$g\left(\frac{2}{(2m+1)\pi}\right) = \sin\left(\frac{(2m+1)\pi}{2}\right) = \pm 1$$

So we have a sequence of points tending to 0 at which g takes the value 0 and another sequence tending to 0 at which g takes the values ± 1. So g has no limit at 0.

Section 5.2

1. If x is non-zero, then there is a sequence $\{\alpha_j\}$ of irrational numbers such that $\alpha_j \to x$. Then $0 = f(\alpha_j) \not\to f(x) = x \ne 0$. So f is discontinuous at x.

Let $\varepsilon > 0$. If $x = 0$ and if $|t - x| < \varepsilon$ then

$$|f(t) - f(x)| = |f(t)| = \begin{cases} |t| & \text{if } t \text{ is rational} \\ 0 & \text{if } t \text{ is irrational} \end{cases}$$

$$\text{so } < \varepsilon.$$

So f is continuous at 0.

3. An isolated point has no neighbors so $\lim_{x \to c} f(x)$ makes no sense.

5. $f(x, y) = x$.

7. If $x = \alpha$ is irrational then any sequence $q_j = \frac{a_j}{b_j}$ of rational number approaching α has the property that $b_j \to \infty$. So $f(q_j) = b_j \to \infty$. Hence f is discontinuous at α.

If q is rational and nonzero, say $q = \frac{a}{b}$ in lowest terms, then there is a sequence $\{\alpha_j\}$ of irrationals approaching q. Hence

$$0 = f(\alpha_j) \not\to f(q) = b.$$

So f is discontinuous at q.

If x is 0 then similar reasoning shows that f is discontinuous at 0.

Section 5.3

1. Let $f(x) = |x - \sqrt{2}/2|$. Then x is positive at every point of $[0,1]$ except $\sqrt{2}/2$. But f is 0 at $\sqrt{2}/2$.

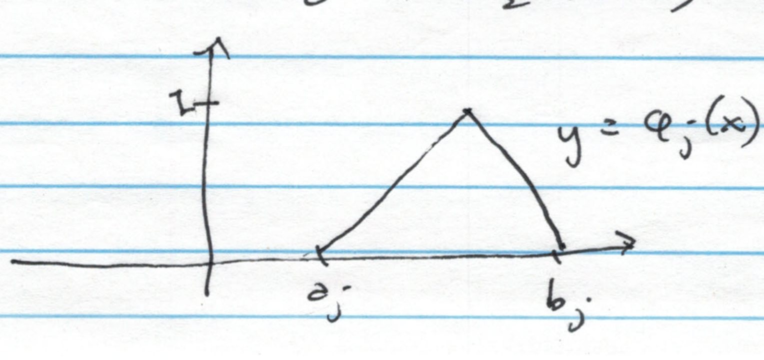
3. Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Then $U = \mathbb{R}$ is open but $f(U) = [0, \infty)$ is not open.

5. Let $f(x) = x^2$. Then $f([-2,-1]) = f([1,2])$. On the other hand, $f^{-1}(C)$ and $f^{-1}(D)$ will be disjoint because if $x \in f^{-1}(C)$ then $f(x) \in C$, $f(x) \notin D$ so $x \notin f^{-1}(D)$.

9. Let $f: (0,1) \to (0,1)$ be given by $f(x) = \frac{x}{2}$.

14. The complement of E is an open set U. Then U is the countable disjoint union of open intervals $I_j = (a_j, b_j)$. Define φ_j on I_j by

$$\varphi_j(x) = \begin{cases} x - a_j, & a_j < x < \frac{a_j + b_j}{2} \\ -\left(x - \frac{a_j+b_j}{2}\right) + 1, & \frac{a_j+b_j}{2} < x < b_j \end{cases}$$



Define
$$f(x) = \begin{cases} \varphi_j(x) & \text{if } x \in I_j \\ 0 & \text{if } x \in E. \end{cases}$$

Section 5.4

1. Let $A = \{a_j\}$ with $a_1 < a_2 < a_3 < \cdots$

Let $f(x) = j$ on the interval $[a_j, a_{j+1})$.

3. Let $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$

Then f^2 is continuous but f is not.

4. If $f(s) \leq f(t)$ when $s \leq t$ and $g(s) \leq g(t)$ when $s \leq t$ then, adding, $f(s) + g(s) \leq f(t) + g(t)$ when $s \leq t$, so $f+g$ is increasing.

Let $f(x) = x$ and $g(x) = 2x$, both increasing. But $f(x) - g(x) = -x$, decreasing.

$f(x) = -e^{-x}$ is increasing, $g(x) = -e^{-x}$ is increasing.

But $f \cdot g(x) = e^{-2x}$ is decreasing.

Also $\frac{1}{f(x)}$ is decreasing.

7. Let α be a point that is a discontinuity of the first kind. So

$$A_1 \equiv \lim_{x \to \alpha^-} f(x) \neq \lim_{x \to \alpha^+} f(x) \equiv A_2.$$

Let q be a rational number between A_1 and A_2. We may choose q for each discontinuity so that they are all different. So we have a univalent map

$$\{\text{discont. of first kind}\} \to \mathbb{Q}.$$

Hence the set of discontinuities of the first kind is countable.

Section 6.1

1. Only for $k=1$. If k is even then

$$f(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x \leq 0 \end{cases}$$

satisfies f^k is differentiable at all x but f is not diff. at 0.

If k is odd and > 1, then $f(x) = x^{1/k}$ satisfies f^k is differentiable at all x but f is not diff. at 0.

6. For $x \neq 0$,

$$f'(x) = \frac{3}{2} x^{1/2} \sin(1/x) - x^{-1/2} \cos(1/x).$$

But

$$f'(0) = \lim_{h \to 0} \frac{h^{3/2} \sin \frac{1}{h} - 0}{h} = 0.$$

$\lim_{x \to 0} f'(x)$ does not exist because of the $x^{-1/2} \cos(1/x)$ term. So f' is discontinuous at 0.

7, The discontinuity must be of the second kind, and it is.

13. a) $$\lim_{h \to 0} \frac{f^2(x+h) - f^2(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot (f(x+h) + f(x))$$

$$= f'(x) \cdot 2 f(x),$$

b) $$[(f+g)^2]' = 2(f+g)(f'+g')$$

$$= 2[ff' + gg' + f'g + fg']$$

c) We have

$$(f^2 + 2fg + g^2)' = 2ff' + 2gg' + 2f'g + 2g'f$$

but $(f^2)' = 2ff'$

$(g^2)' = 2gg'$

Hence

$$(2fg)' = 2f'g + 2g'f$$

or $(fg)' = f'g + g'f.$

Section 6.2

2. $$f(b) = f(a) + (f(b) - f(a))$$

$$= f(a) + (b-a)\cdot f'(\xi)$$

for some $a < \xi < b$.

Hence

$$f(b) \le m + (b-a)K$$

4. It is easier to check that

$$\log x < C_\alpha x^\alpha$$

for $x \geq 1$. Note that $0 = \log 1 < 1^\alpha = 1$.

Also

$$1/x = (\log x)' \leq \frac{1}{\alpha}\alpha x^{\alpha-1} = \frac{1}{\alpha}(x^\alpha)'$$

Because $1 \leq \frac{1}{\alpha}\cdot\alpha\cdot x^\alpha$ for $x \geq 1$.

So x^α starts off at 1 greater than $\log 1$ and increases more quickly.

Now if $t \leq 1$ then $\frac{1}{t} \geq 1$ so

$$\log\frac{1}{t} \leq \frac{1}{\alpha}\left(\frac{1}{t}\right)^\alpha$$

$$\text{or } -\log t \leq \frac{1}{\alpha}t^{-\alpha}$$

$$\text{or } |\log t| \leq \frac{1}{\alpha}t^{-\alpha}$$

which is what we want.

5. Let $f(x) = x$ on $(-1, 1)$. Then

$$f^2(x) = x^2 \text{ and } \underset{2x}{(f^2)'} < 0 \text{ for } x < 0.$$

If we assume $f(x) > 0 \ \forall x$ then everything is OK.

8. Let $f(x) = x^{1/3}$. Then we are studying

$$f(x^4+1) - f(x^4).$$ By the mean value theorem, this equals

$$\left[(x^4+1) - x^4\right] \cdot f'(\xi) \qquad (*)$$

for some $x^4 < \xi < x^4 + 1$.

$$\text{Now } (*) = 1 \cdot \frac{1}{3} \xi^{-2/3}$$

$$\leq \frac{1}{3}(x^4)^{-2/3} = \frac{x^{-8/3}}{3} \to 0$$

as $x \to +\infty$.