Math 4111

SOLUTIONS TO HW 7

§7.5 #1. $\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx$

$$\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 \frac{\left(x - \frac{x^3}{3!} + - \cdots\right)}{x}\,dx = \int_0^1 1 - \frac{x^2}{3!} + - \cdots dx$$

and this integral exists.

$$\int_1^\infty \frac{\sin x}{x}\,dx = \lim_{N\to\infty} \int_1^N \frac{\sin x}{x}\,dx = \lim_{N\to\infty} \left.\frac{-\cos x}{x}\right|_1^N + \int_1^N \frac{\cos x}{x^2}\,dx$$

$$= \cos 1 + \lim_{N\to\infty} \int_1^N \frac{\cos x}{x^2}\,dx$$

and this last integral clearly exists.

2. On any interval $[\;,\;]$ where $f' > 0$,

$$\int_h^k |f'(x)|\,dx = \int_h^k f'(x)\,dx = f(k) - f(h)$$
$$= |f(k) - f(h)|.$$

On any interval $[c,d]$ where $f' < 0$,

$$\int_c^d |f'(x)|\,dx = \int_c^d - f'(x)\,dx = -\,[f(d) - f(c)]$$
$$= f(c) - f(d) = |f(d) - f(c)|.$$

Thus $\int_a^b |f'(x)|\,dx = \sum |f(s_j) - f(s_{j-1})|$

and this defines VF.

5. First consider φ linear: $\varphi(t) = at + b$.
Then

$$\varphi\left(\int_0^1 f(x)\,dx\right) = a\left(\int_0^1 f(x)\,dx\right) + b$$

$$= \int_0^1 af(x) + b\,dx = \int_0^1 \varphi(f(x))\,dx. \quad (*)$$

Now, if ψ is convex, then ψ is the sup of linear functions (functions that describe the tangent lines to the graph). Say

$\psi = \sup \varphi$, φ linear.

From $(*)$,

$$\varphi\left(\int_0^1 f(x)\,dx\right) \le \int_0^1 \psi(f(x))\,dx$$

Now take the sup over φ linear on the left to get

$$\psi\left(\int_0^1 f(x)\,dx\right) \le \int_0^1 \psi(f(x))\,dx$$

§8.1 #4. Work on the interval $[-\pi, \pi]$.

Let
$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin jx \, dx$$
$$= \frac{1}{\pi}\left[\frac{x(-\cos jx)}{j}\Big]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos jx}{j} dx \right]$$
$$= \frac{1}{\pi}\left[\left(\frac{\pi(-(-1)^j)}{j} + \frac{(-\pi)((-1)^j)}{j}\right) + \frac{\sin jx}{j}\right]_{-\pi}^{\pi}$$
$$= \frac{2}{j}(-1)^{j+1}.$$

Then, by the theory of Fourier series,
$$\sum_{j=1}^{\infty} \frac{2}{j}(-1)^{j+1} \sin jx = x$$

or
$$\sum_{j=1}^{N} \frac{2}{j}(-1)^{j+1} \sin jx \to x \quad \text{as } N \to \infty.$$

7. $$\sin x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!}$$

8. Let $\varepsilon > 0$. Choose $N > \frac{1}{\varepsilon}$. If $J, K > N$,

then
$$\left| \sum_{j=J}^{K} \frac{\sin jx}{j^2} \right| \le \sum_{j=J}^{K} \frac{1}{j^2} \le C \cdot \int_J^K \frac{1}{x^2} dx$$
$$= -\frac{C}{x}\Big]_J^K = \frac{C}{J} - \frac{C}{K} \le \frac{C}{J} < C\varepsilon.$$

§8.2 #1. If $\sum_{j=1}^{\infty} f_j$ converges uniformly, then the sequence $\left\{\sum_{j=1}^{N} f_j\right\}_{N=1}^{\infty}$ converges uniformly to some function f. Let $g_N = \sum_{j=1}^{N} f_j$. Then

$g_N \to f$ uniformly. But then f is continuous because each g_N is.

2. Let $f_j(x) = x + \frac{1}{j}$. Then $\lim_{j\to\infty} f_j(x) = x \equiv f_0(x)$ uniformly because

$$|f_j(x) - f_0(x)| = \frac{1}{j} \to 0 \text{ independent of } x.$$

But $f_j^2 = x^2 + \frac{2x}{j} + \frac{1}{j^2} \to x^2 \equiv g_0$.

However $|f_j^2 - g_0| = \left|\frac{2x}{j} + \frac{1}{j^2}\right| > \frac{2x}{j}$ for $x > 0$

which, if $x > j^2$, is greater than $2j$.

So convergence not uniform.

If we assume the f_j are uniformly bounded, $|f_j(x)| \le M \ \forall x$, then everything is OK.

4. If $\{f_j\}$ converges pointwise to f_0, then

$$S_N = f_1 + \sum_{j=2}^{N} (f_j - f_{j-1})$$

$= f_N$ and this converges to f_0.

Conversely, if $f_1 + \sum_{j=2}^{\infty} f_j$ converges pointwise to f_0 then f_N converges pointwise to f_0.

The same argument works for uniform convergence.