Math 4111 Fall, 2020
Krantz

PRACTICE FIRST MIDTERM EXAM

General Instructions: Read the statement of each problem carefully. If you want full credit on a problem then you must show your work. If you only write the answer then you will *not* receive full credit.

Be sure to ask questions if anything is unclear. This exam has 8 questions and is worth 100 points. You will have 50 minutes to take this exam.

(8 points) **1.** Prove that the sequence

$$a_j = \frac{(-1)^{3j} \cdot 4}{j^2}$$

converges to 0.

Let $\varepsilon > 0$. Let N be greater than $\frac{2}{\sqrt{\varepsilon}}$ and let $j > N$. Then

$$|a_j - 0| = \left|\frac{(-1)^{3j} \cdot 4}{j^2}\right| = \frac{4}{j^2} < \frac{4}{N^2} < \frac{4\varepsilon}{4} = \varepsilon.$$

So $a_j \to 0$.

(8 points) **2.** What is the multiplicative inverse of the complex number $3 - 7i$?

Mult. inverse is $\frac{3+7i}{3^2+7^2} = \frac{3+7i}{58} = \frac{3}{58} + \frac{7}{58} i$

(8 points) **3.** Find all cube roots of the complex number $1 - i$.

$1 - i = \sqrt{2}\, e^{i 7\pi/4}$

$(re^{i\theta})^3 = \sqrt{2}\, e^{i 7\pi/4} \Rightarrow r = 2^{1/6}, \theta = \frac{7\pi}{12}$. $z_1 = 2^{1/6} e^{i \frac{7\pi}{12}}$

$(re^{i\theta})^3 = \sqrt{2}\, e^{i(7\pi/4 + 2\pi)} \Rightarrow r = 2^{1/6}, \theta = \frac{15\pi}{12}$. $z_2 = 2^{1/6} e^{i 15\pi/12}$

$(re^{i\theta})^3 = \sqrt{2}\, e^{i(7\pi/4 + 4\pi)} \Rightarrow r = 2^{1/6}, \theta = \frac{23\pi}{12}$. $z_3 = 2^{1/6} e^{i 23\pi/12}$

(8 points) **4.** What is the least upper bound of the set $\{x \in \mathbb{R} : x^2 < 11\}$? Give a reason for your answer.

The l.u.b is $\sqrt{11} \in \mathbb{R}$.

Any number $x < \sqrt{11}$ satisfies $x^2 < 11$ so is not a l.u.b.

Any number $x > \sqrt{11}$ satisfies $x^2 > 11$ so is not a l.u.b.

(8 points) **5.** Prove that if

$$\liminf_{j\to+\infty} a_j = \limsup_{j\to+\infty} a_j ,$$

then the sequence $\{a_j\}$ converges.

If a_{j_k} is any subsequence then

$$\liminf_{j\to\infty} a_j \le \liminf_{k\to\infty} a_{j_k} \le \limsup_{k\to\infty} a_{j_k} \le \limsup_{j\to\infty} a_j .$$

The left and right are equal hence the middle two are equal, So every subsequence converges to the same limit. That gives the result.

(9 points) **6.** Discuss convergence and divergence for each of these series.

(a) $\sum_{j=1}^{\infty} \frac{1}{j^2}$

(b) $\sum_{j=1}^{\infty} \frac{j!}{(5j)!}$

(c) $\sum_{j=2}^{\infty} \frac{1}{j \cdot \log^{1.5} j}$

a) converge by the integral test

b) $\frac{j!}{(5j)!} \leq \frac{j \cdot (j-1) \cdots 2 \cdot 1}{5j \cdot (5j-1) \cdots 4j} < \left(\frac{1}{5}\right)^j$ and $\sum \left(\frac{1}{5}\right)^j$ converges.

c) converges by integral test.

(9 points) **7.** Let $\sum_j a_j$ and $\sum_j b_j$ be series of positive terms. Prove that, if there is a constant $C > 0$ such that

$$\frac{1}{C} a_j \leq b_j \leq C a_j$$

for all j large, then either both series converge or both series diverge.

We see that $a_j \leq C b_j$. If $\sum b_j$ converges then $\sum a_j$ converges by comparison test.

We also see that $b_j \leq C a_j$. If $\sum a_j$ converges then $\sum b_j$ converges by comparison test.

(9 points) **8.** Discuss convergence or divergence of the series

$$\sum_{j=1}^{\infty} \frac{\sin^2 j}{j}.$$

$$\sum_{j=1}^{\infty} \frac{\sin^2 j}{j} = \sum_{j=1}^{\infty} \frac{\frac{1-\cos 2j}{2}}{j}$$

$$= \sum_{j=1}^{\infty} \frac{1}{2j} - \sum_{j=1}^{\infty} \frac{\cos 2j}{2j} = A + B$$

B converges by an argument similar to the one in the book. But A diverges.

Hence $\sum_{j=1}^{\infty} \frac{\sin^2 j}{j}$ diverges.

(8 points) **9.** Let $\sum_j a_j$ and $\sum_j b_j$ be convergent series of positive real numbers. Discuss the convergence of $\sum_j (a_j \cdot b_j^2)$.

When j is large, $a_j < 1$. So

$$a_j b_j^2 < b_j^2.$$

Also for j large $b_j < 1$. So

$$b_j^2 < b_j$$

So $\sum a_j b_j^2$ converges by the comparison test.

(9 points) **10.** If $1/2 > b_j > 0$ and if $\sum_j b_j$ converges then prove that

$$\sum_{j=1}^{\infty} \frac{b_j}{1+b_j}$$

converges.

Clearly $1 < 1 + b_j < \frac{3}{2}$.

So $\frac{2}{3} b_j < \frac{b_j}{1+b_j} < b_j$.

Series converges by comparison test.

(8 points) **11.** Let $\gamma > 0$ be a fixed real number. Give an example of a set whose sup and inf differ by γ.

Let $S = [0, \gamma]$.

(8 points) **12.** Consider $\{a_j\}$ both as a sequence and as a set. How are the limsup of the sequence and the sup of the set related? How are the liminf of the sequence and the inf of the set related? Give an example where they are both the same. Give an example where they are both different.

Let $\{a_j\} = \{\frac{1}{j}\}$. Then

$\sup\{a_j\} = 1$ but $\limsup\{a_j\} = 0$.

Let $\{a_j\} = \{-\frac{1}{j}\}$. The

$\inf\{a_j\} = -1$ but $\liminf\{a_j\} = 0$.

Let $\{a_j\} = \{1\}$.

Then $\sup\{a_j\} = \inf\{a_j\} = \limsup\{a_j\} = \liminf\{a_j\} = 1$