# SOLUTIONS TO MIDTERM 2

1. Let $x \in \{x \in \mathbb{R} : 1 < x^2 < 2\} = \{x \in \mathbb{R} : 1 < |x| < \sqrt{2}\}$. Let $\varepsilon = \min\{|x| - 1, \sqrt{2} - |x|\}$. If $t \in (x - \varepsilon, x + \varepsilon)$, then $1 < |t| < \sqrt{2}$. So $t \in \{x \in \mathbb{R} : 1 < |t| < \sqrt{2}\}$. Hence $(x - \varepsilon, x + \varepsilon) \subset \{x \in \mathbb{R} : 1 < |x| < \sqrt{2}\}$. So the set is open.

2. Let $O_j = \mathbb{R} \setminus E_j$. Then $O_j$ is open. Let $\mathcal{O} = \bigcup_{j=1}^{\infty} O_j$. If $x \in \mathcal{O}$, then $x \in O_j$ for some $j$. So $\exists\ \varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq O_j$. Hence $(x - \varepsilon, x + \varepsilon) \subseteq \mathcal{O}$. So $\mathcal{O}$ is open. But then $^c\mathcal{O} = {}^c\left(\bigcup_{j=1}^{\infty} O_j\right) = \bigcap_{j=1}^{\infty} {}^cO_j = \bigcap_{j=1}^{\infty} E_j$ is closed.

3. Let $E$ and $F$ be compact. Then both sets are closed and both sets are bounded. So $E \cap F$ is closed and bounded. So $E \cap F$ is compact.

4. Let $C$ be the Cantor set. Let $x \in C$. Since $C$ has $0$ length, any interval $(x - \varepsilon, x + \varepsilon)$ will intersect the complement of $C$. So $C$ has no interior. Likewise, any interval $(x - \varepsilon, x + \varepsilon)$ will intersect $^cC$ and will intersect $C$ itself (since $x \in C$). So every point of $C$ is in the boundary.

Since $C$ is closed, no point of the complement is in the boundary

5. a) The left and right limits exist but do not agree with $f(0)$. So this is a discontinuity of the first kind.

b) The right limit does not exist. This is a discontinuity of the second kind.

c) The left and right limits exist but do not agree with the value at 0. So this is a discontinuity of the first kind.

6. Use l'Hôpital's rule:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \frac{1}{1} = 1.$$

7. $f(3) - f(0) = f'(\xi)\cdot(3-0)$ for some $0<\xi<3$.

So

$$|f(3) - f(0)| = 3\,|f'(\xi)| \le 3$$

$$|f(3)| \le 3.$$

8. Let

$$f(x) = \begin{cases} x^{3/2}\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

If $x \neq 0$, then $f'(x) = \frac{3}{2}x^{\frac{1}{2}}\sin\frac{1}{x} - x^{-\frac{1}{2}}\cos\frac{1}{x}$.

But

$$f'(0) = \lim_{h \to 0} \frac{h^{3/2}\sin\frac{1}{h} - 0}{h} = \lim_{h \to 0} h^{\frac{1}{2}}\sin\frac{1}{h} = 0.$$

We see that $\lim_{x \to 0} f'(x)$ does not exist because of the $-x^{-\frac{1}{2}}\cos\frac{1}{x}$ term. So $f'$ has a discontinuity of the second kind at $0$.

9. Let $f(x) = \sqrt{x}$. So we are studying

$$f(x+1) - f(x) = f'(\xi)\cdot(1-0)$$

for some $x < \xi < x+1$.

$$\text{But } f'(\xi) = \frac{1}{2}\xi^{-1/2} < \frac{1}{2}x^{-1/2}.$$

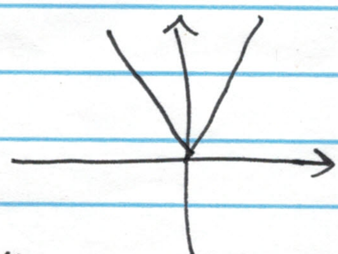
So $f(x+1) - f(x) \leq \frac{1}{2}x^{-1/2}\cdot 1 \to 0$ as $x \to +\infty$.

10. $$f(x) = \begin{cases} x^3 & \text{if } x \geq 0 \\ -x^3 & \text{if } x < 0. \end{cases}$$

So $f'(x) = \begin{cases} 3x^2, & x \geq 0 \\ -3x^2, & x < 0 \end{cases}$

$f''(x) = \begin{cases} 6x, & x \geq 0 \\ -6x, & x < 0. \end{cases}$

The graph of $f''$ looks like this:



So $f''$ is <u>not</u> differentiable at 0.

Thus $f$ is $C^2$ but not $C^3$. Also $f''$ is Lipschitz 1 with Lipschitz constant 6.

So $f$ is $C^{2,1}$.

11. Let $f(x) = x^{1/x}$. Then let $g(x) = \log f(x) = \frac{1}{x} \log x$.

Write
$$g(x) = \frac{\log x}{x}.$$

Then both numerator and denominator tend to $+\infty$ so l'Hôpital's rule applies.

$$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} \frac{1/x}{1} = 0.$$

Thus $\lim_{x \to +\infty} f(x) = 1.$

12. Let $x, y \in \mathbb{R}$. Let $k \in K$. Then

$$y - k = (y-x) + (x-k)$$

So $$|y-k| \leq |y-x| + |x-k|$$

Hence
$$\inf_{k \in K} |y-k| \leq |y-x| + |x-k|$$

$$f(y) \leq |y-x| + |x-k|.$$

This is true $\forall k \in K$. So

$$f(y) \leq |y-x| + \inf_{k \in K} |x-k|$$

$$f(y) \leq |y-x| + f(x)$$

$$f(y) - f(x) \leq |y-x|.$$

Likewise
$$f(x) - f(y) \leq |y-x|.$$

So $$|f(y) - f(x)| \leq |y-x|.$$