SOLUTIONS TO MIDTERM 2

1. $\underline{\underline{r}}'(t) = 2t\,\underline{\underline{i}} - \underline{\underline{j}} - 2t\,\underline{\underline{k}}$

$\underline{\underline{r}}'(1) = 2\underline{\underline{i}} - \underline{\underline{j}} - 2\underline{\underline{k}}$

$\underline{\underline{r}}''(t) = 2\underline{\underline{i}} + 0\underline{\underline{j}} - 2\underline{\underline{k}}$

$\underline{\underline{r}}''(1) = 2\underline{\underline{i}} - 2\underline{\underline{k}}.$

$$\underline{\underline{r}}'(1) \times \underline{\underline{r}}''(1) = \det \begin{pmatrix} \underline{\underline{i}} & \underline{\underline{j}} & \underline{\underline{k}} \\ 2 & -1 & -2 \\ 2 & 0 & -2 \end{pmatrix}$$

$$= \underline{\underline{i}}(2) - \underline{\underline{j}}(0) + \underline{\underline{k}}(2) = 2\underline{\underline{i}} + 2\underline{\underline{k}}$$

$\|\underline{\underline{r}}'(1) \times \underline{\underline{r}}''(1)\| = \sqrt{8} = 2\sqrt{2}$

$\|\underline{\underline{r}}'(1)\| = \sqrt{4+1+4} = 3$

$$\kappa = \frac{\|\underline{\underline{r}}'(1) \times \underline{\underline{r}}''(1)\|}{\|\underline{\underline{r}}'(1)\|^3} = \frac{2\sqrt{2}}{3^3} = \frac{2\sqrt{2}}{27} \qquad c$$

2. $\underline{\underline{r}}'(t) = -\sin t\,\underline{\underline{i}} + \cos t\,\underline{\underline{j}} + 3\underline{\underline{k}}$

$$\underline{\underline{T}}(t) = -\frac{\sin t}{\sqrt{10}}\underline{\underline{i}} + \frac{\cos t}{\sqrt{10}}\underline{\underline{j}} + \frac{3}{\sqrt{10}}\underline{\underline{k}}$$

$$\underline{\underline{T}}'(t) = -\frac{\cos t}{\sqrt{10}}\underline{\underline{i}} \; \frac{-\sin t}{\sqrt{10}}\underline{\underline{j}} + 0\underline{\underline{k}}$$

$\underline{\underline{N}}(t) = -\cos t\,\underline{\underline{i}} - \sin t\,\underline{\underline{j}}$

$\underline{\underline{N}}\left(\frac{\pi}{2}\right) = -\underline{\underline{j}}.$

②

$$\underline{\underline{r}}''(t) = -\cos t\,\underline{\underline{i}} - \sin t\,\underline{\underline{j}} + 0\,\underline{\underline{k}}$$

$$\underline{\underline{r}}''\left(\tfrac{\pi}{2}\right) = 0\,i - 1\,\underline{\underline{j}} + 0\,\underline{\underline{k}}$$

$$\underline{\underline{r}}'\left(\tfrac{\pi}{2}\right) \times \underline{\underline{r}}''\left(\tfrac{\pi}{2}\right) = \det\begin{pmatrix} \underline{\underline{i}} & \underline{\underline{j}} & \underline{\underline{k}} \\ -1 & 0 & 3 \\ 0 & -1 & 0 \end{pmatrix}$$

$$= \underline{\underline{i}}\cdot 3 - \underline{\underline{j}}\cdot 0 + \underline{\underline{k}}\,1$$

$$\left\|\underline{\underline{r}}'\left(\tfrac{\pi}{2}\right) \times \underline{\underline{r}}''\left(\tfrac{\pi}{2}\right)\right\| = \sqrt{10}$$

$$\left\|\underline{\underline{r}}'\left(\tfrac{\pi}{2}\right)\right\| = \sqrt{10}$$

$$k = \frac{\left\|\underline{\underline{r}}'\left(\tfrac{\pi}{2}\right) \times \underline{\underline{r}}''\left(\tfrac{\pi}{2}\right)\right\|}{\left\|\underline{\underline{r}}'\left(\tfrac{\pi}{2}\right)\right\|^3} = \frac{\sqrt{10}}{(\sqrt{10})^3} = \frac{1}{10}.$$

radius of curvature $= \rho = \frac{1}{k} = 10$

centre $= P + \rho\,\underline{\underline{N}} = \left(0, 1, \frac{3\pi}{2}\right) + 10\langle 0, -1, 0\rangle$

$$= \left(0, -9, \frac{3\pi}{2}\right).$$

e

3. $\underline{r}'(t) = 2t\,\underline{i} - 3t^2\,\underline{j} + 1\,\underline{k}$

$v(t) = \sqrt{4t^2 + 9t^4 + 1}$

$\frac{dv}{dt} = \frac{1}{2}(4t^2 + 9t^4 + 1)^{-1/2} \cdot (8t + 36t^3)$

$\frac{dv}{dt}(1) = \frac{1}{2}(14)^{-1/2} \cdot 44 = \frac{22}{\sqrt{14}} = a_T(1).$

$\underline{r}''(t) = 2\,\underline{i} - 6t\,\underline{j} + 0\,\underline{k}$

$\underline{a}(1) = \underline{r}(1) = 2\,\underline{i} - 6\,\underline{j} + 0\,\underline{k}$

$a_N = \sqrt{\|\underline{a}\|^2 - a_T^{\,2}}$

$= \sqrt{40 - \frac{484}{14}} = \sqrt{\frac{560 - 484}{14}} = \frac{\sqrt{76}}{\sqrt{14}}$

$= \frac{\sqrt{38}}{\sqrt{7}}.$

b

4. $\frac{30^2}{(800{,}000)^3} = \frac{p^2}{(2{,}000)^3}$

$9 \cdot 10^2 \cdot 8^{-3} \cdot 10^{-15} = p^2 \cdot 2^{-3} \cdot 10^{-9}$

$9 \cdot 2^3 \cdot 8^{-3} \cdot 10^{-4} = p^2$

$\frac{9}{64} \cdot 10^{-4} = p^2$

$\frac{3}{8} \cdot \frac{1}{100} = p$

d

4

5. $\frac{\partial f}{\partial y} = -\sin(xy) - xy\cos(xy)$

$\frac{\partial^2 f}{\partial x \partial y} = -y\cos(xy) - y\cos(xy) + xy^2\sin(xy)$

$= -2y\cos(xy) + xy^2\sin(xy)$

a

6. $D_{\underline{u}} f(1,0) = \nabla f(1,0) \cdot \underline{u}$

$= \langle y^3 - 2xye^{x^2y}, 3xy^2 - x^2e^{x^2y} \rangle \Big|_{(1,0)} \cdot \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

$= \langle 0, -1 \rangle \cdot \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

$= \frac{1}{\sqrt{2}}.$

d

7. The level set $f = 1$ is

$1 = \frac{2x^2}{x^2+y^2}$

$x^2 + y^2 = 2x^2$

b

$x^2 - y^2 = 0$

$(x-y)(x+y) = 0$ two lines

8. a

9. $|f(x,y)| \le \dfrac{\frac{1}{2}x^4+\frac{1}{2}y^4}{x^2+y^2} \to 0$

So f is continuous at $(0,0)$. c

10. $\dfrac{\partial z}{\partial t} = \dfrac{\partial f}{\partial x}\dfrac{\partial y}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t}$

$= (y^2 - y\cos(xy))\cdot(-2t) + (2xy - x\cos(xy))(2)$

$=$

When $s=1$, $t=2$, $x=-3$, $y=5$.

So $\dfrac{\partial z}{\partial t} = (25 - 5\cos 15)\cdot(-4) + (-30 + 3\cos 15)(2)$

$= -160 + 26\cos 15$ d

11. $f_x = -\sin(x+y^2)$

$f_y = -2y\sin(x+y^2)$

$f_{xx} = -\cos(x+y^2)$

$f_{yy} = -2\sin(x+y^2) - 4y^2\cos(x+y^2)$

$f_{xy} = -2y\cos(x+y^2)$

$f_{xxy} = +2y\sin(x+y^2)$

$f_{xyy} = -2\cos(x+y^2) + 4y^2\sin(x+y^2)$

$f_{yyy} = -4y\cos(x+y^2) - 8y\cos(x+y^2) + 8y^3\sin(x+y^2)$

6

$$f_{xxx} = \sin(x+y^2)$$

$$f(h,k) = f(0,0) + f_x(0,0)\,h + f_y(0,0)\,k$$
$$+ \frac{1}{2}\left[f_{xx}(0,0)h^2 + f_{yy}(0,0)k^2 + 2f_{xy}(0,0)hk\right]$$
$$+ \frac{1}{3!}\left[f_{xxx}(0,0)h^3 + f_{xxy}(0,0)h^2k + f_{xyy}(0,0)hk^2\right.$$
$$\left. + f_{yyy}(0,0)k^3\right] + \text{error}$$

$$= 1 + 0h + 0k + \frac{1}{2}\left[-1\cdot h^2 + 0k^2 + 0hk\right]$$
$$+ \frac{1}{3!}\left[0h^3 + 3\cdot 0h^2k + 3(-2)hk^2\right.$$
$$\left. + 0k^3\right] + \text{erra}$$
$$= 1 - \frac{h^2}{2} + \frac{-6}{3!}hk^2 + \text{error}.$$

c

12 $\nabla f = \langle 3x^2, 4y^3 \rangle$

$\nabla f(2,1) = \langle 12, 4 \rangle$

dir. is $= \left\langle \frac{12}{\sqrt{160}}, \frac{4}{\sqrt{160}} \right\rangle$

$= \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle.$

b

13. $z - 2x^2 - y^2 = 0$

gradient is $\langle -4x, -2y, 1 \rangle$

$\langle -8, 4, 1 \rangle$

$(\langle x, y, z \rangle - \langle 2, 2, 4 \rangle) \cdot \langle -8, 4, 1 \rangle = 0$

$\langle x-2, y-2, z-4 \rangle \cdot \langle -8, 4, 1 \rangle = 0$

$(-8x + 16) + (4y - 8) + (z - 4) = 0$

$-8x + 4y + z = -4$

$8x - 4y - z = 4$ e

14. $f_x = \frac{1}{2}(x+y)^{-1/2}$

$f_y = \frac{1}{2}(x+y)^{-1/2}$

$f(1.9, 2.2) \approx f(2, 2) + f_x(2, 2)(-.1) + f_y(2, 2)(+.2)$

$= 2 + \frac{1}{4}(-.1) + \frac{1}{4}(+.2)$

$= 2 + \frac{1}{4}(.1) = 2.025$ a

15. $\nabla f = \langle 3x^2 - y, -3y^2 - x \rangle$.

$\langle 0, 0 \rangle = \langle 3x^2 - y, -3y^2 - x \rangle$

$3x^2 = y$

$3y^2 = -x$

$x^2 = 9y^4$

$3(9y^4) = y$

$y(27y^3 - 1) = 0 \Rightarrow y = 0,\ y = \frac{1}{3}$

$y = \frac{1}{3} \Rightarrow x = -\frac{1}{3}$

$y = 0 \Rightarrow x = 0$

So $(0,0)$ & $(-\frac{1}{3}, \frac{1}{3})$

$$\text{Discrim} = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$$= \det \begin{pmatrix} 6x & -1 \\ -1 & -6y \end{pmatrix} = -36xy - 1$$

$\text{Discrim}(0,0) = -1$ so point is saddle

$\text{Discrim}(-\frac{1}{3}, \frac{1}{3}) = -36(-\frac{1}{3})(\frac{1}{3}) - 1 = 4 - 1 = 3$

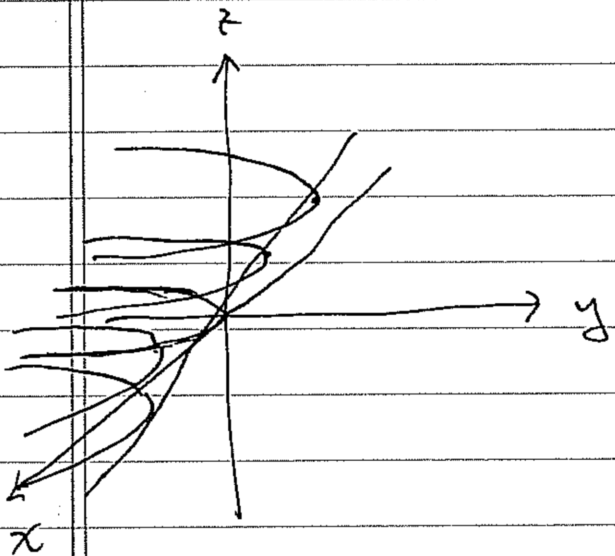
$6x = -2 < 0$, $-6y = -2 < 0$.

So point is a local maximum.

e

16. $z = x^2 + y$

z	level set
-2	$y = -x^2 - 2$
-1	$y = -x^2 - 1$
0	$y = -x^2$
1	$y = -x^2 + 1$
2	$y = -x^2 + 2$



17. $\nabla f = \lambda \nabla g$

$\langle 1, 1 \rangle = \lambda \langle 2x, 4y \rangle \Rightarrow$ $1 = 2\lambda x$, $1 = 4\lambda y$

$x = \frac{1}{2\lambda}$

$y = \frac{1}{4\lambda}$

$\left(\frac{1}{2\lambda}\right)^2 + 2\left(\frac{1}{4\lambda}\right)^2 = 4$

$\frac{1}{4\lambda^2} + \frac{1}{8\lambda^2} = 4$

$2 + 1 = 32\lambda^2$

$\lambda = \pm\sqrt{\frac{3}{32}}$

$x = \frac{1}{2\frac{\sqrt{3}}{4\sqrt{2}}} = \frac{2\sqrt{2}}{\sqrt{3}}$, $y = \frac{\sqrt{2}}{\sqrt{3}}$ maximum (value is $\sqrt{6}$)

$x = \frac{-2\sqrt{2}}{\sqrt{3}}$, $y = \frac{-\sqrt{2}}{\sqrt{3}}$ minimum (value is $-\sqrt{6}$)