Symplectic groupoids of log symplectic manifolds

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Blow-up of Lie groupoids

Theorem 1 *[1] Let $\mathcal{H} \rightrightarrows D$ be a closed Lie subgroupoid of $\mathcal{G} \rightrightarrows M$ over the closed hypersurface D, and define*

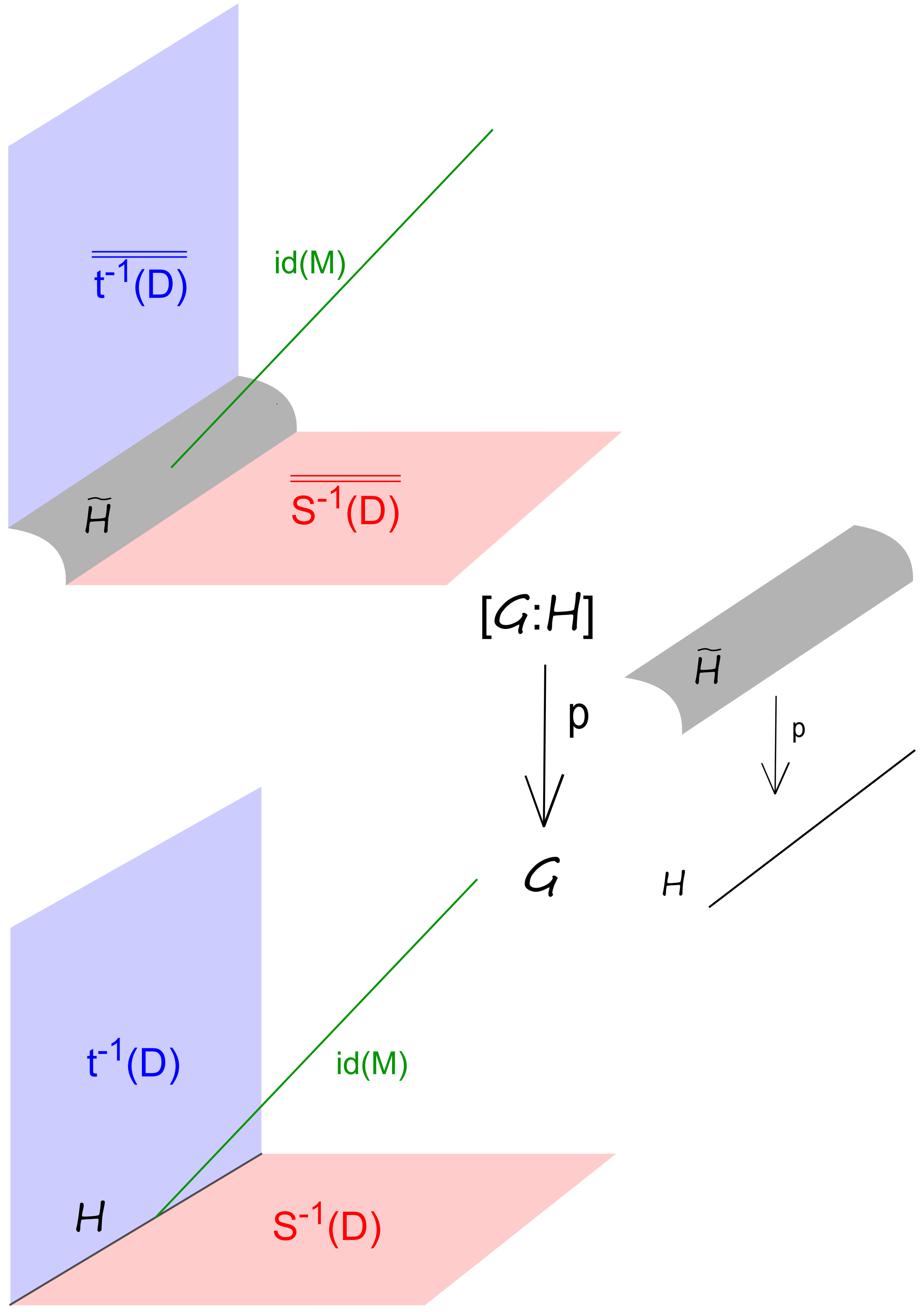
$$[\mathcal{G}:\mathcal{H}] = \mathsf{Bl}_{\mathcal{H}}(\mathcal{G}) \backslash (\overline{\overline{s^{-1}(D)}} \cup \overline{\overline{t^{-1}(D)}}), \qquad (1)$$

where $\overline{\overline{s^{-1}(D)}}$ (resp. $\overline{\overline{t^{-1}(D)}}$) is the proper transform of $s^{-1}(D)$ (resp. $t^{-1}(D)$). There is a unique Lie groupoid structure $[\mathcal{G}:\mathcal{H}] \rightrightarrows M$ such that the blow-down map restricts to a base-preserving Lie groupoid morphism

$$p : [\mathcal{G}:\mathcal{H}] \to \mathcal{G}.$$

The Lie algebroid $\mathbf{Lie}([\mathcal{G}:\mathcal{H}])$ is the elementary modification of $\mathbf{Lie}(\mathcal{G})$ along $\mathbf{Lie}(\mathcal{H})$, i.e. $\mathbf{Lie}([\mathcal{G}:\mathcal{H}])$ has sheaf of sections defined by

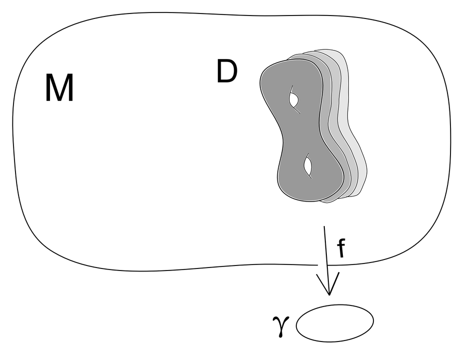
$$\mathbf{Lie}([\mathcal{G}:\mathcal{H}]) = \{X \in \mathbf{Lie}(\mathcal{G}) \mid X|_D \in \mathbf{Lie}(\mathcal{H})\}.$$



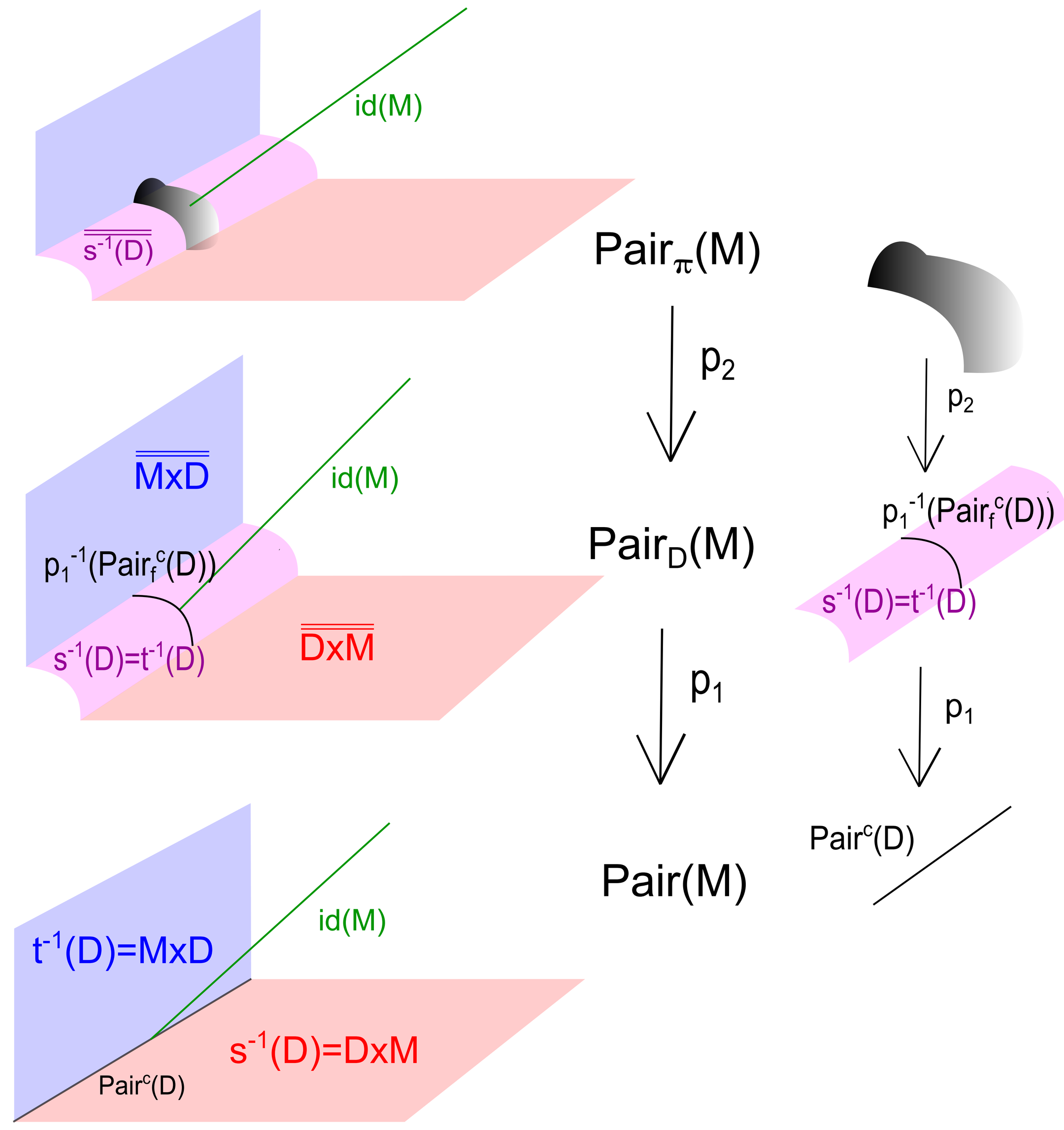
Birational construction

Definition 2 *A* log symplectic manifold *is a $2n$-manifold M with a Poisson structure π whose Pfaffian, π^n, vanishes transversely. Moreover, (M,π) is* proper *if each connected component D_j of the degeneracy locus D is compact and contains a compact symplectic leaf.*

Theorem 3 *[2] For a proper log symplectic manifold, each D_j is a symplectic mapping torus. In particular, $f_j : D_j \to \gamma_j$ is a symplectic fibre bundle.*



A 2-stage blow-up of the pair groupoid $\mathrm{Pair}(M)$, where $\mathrm{Pair}^c(D) = \coprod_j (D_j \times D_j)$ and $\mathrm{Pair}^c_f(D) = \coprod_j (D_j \times_{\gamma_j} D_j)$, yields the symplectic pair groupoid $\mathrm{Pair}_\pi(M)$.



Theorem 4 *[1] For a proper log symplectic manifold (M,π), the symplectic pair groupoid $\mathrm{Pair}_\pi(M)$ is the adjoint symplectic groupoid. That is, if $\mathcal{G} \rightrightarrows M$ is a symplectic groupoid, then there exists a groupoid morphism $\varphi : \mathcal{G} \to \mathrm{Pair}_\pi(M)$.*

Gluing construction

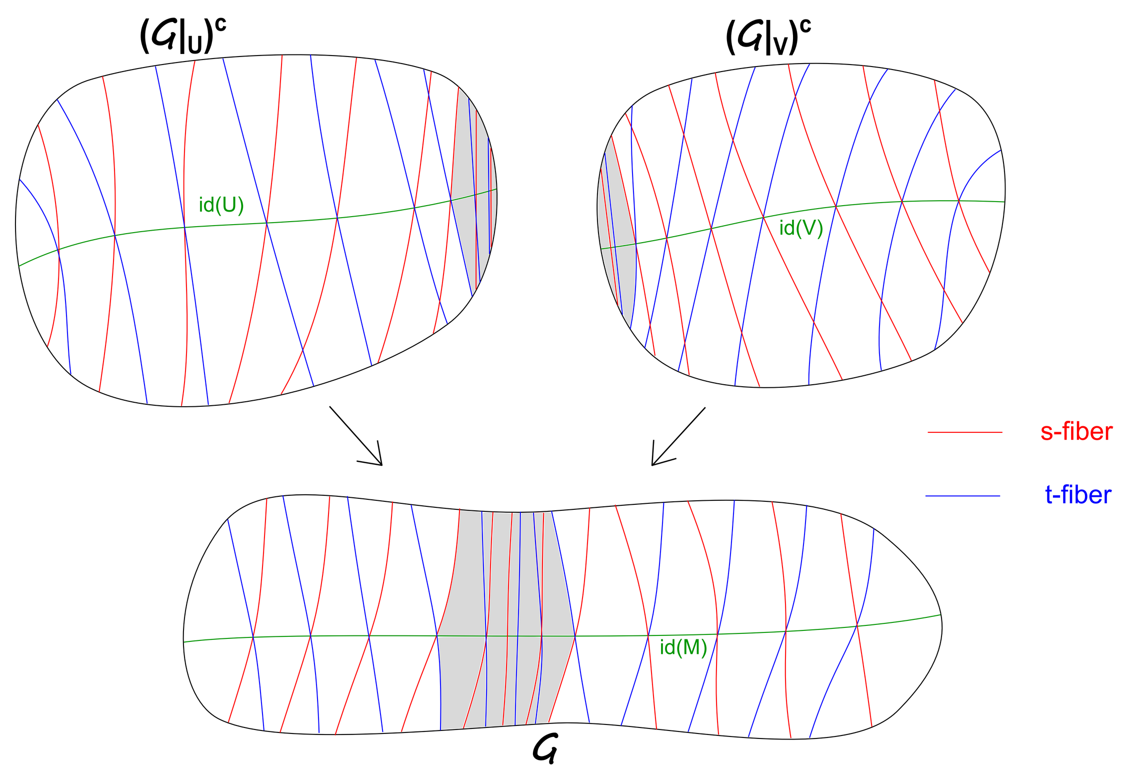
The restriction of a Lie groupoid $\mathcal{G} \rightrightarrows M$ to an open set $U \subset M$, denoted by $(\mathcal{G}|_U)^c$, is the source-connected part of $s^{-1}(U) \cap t^{-1}(U)$.

An *orbit cover* of $\mathcal{G} \rightrightarrows M$ is a locally finite cover $\{U_i\}_{i\in\mathtt{I}}$ of M such that each orbit of $\mathcal{G} \rightrightarrows M$ is contained in U_i for some $i \in \mathtt{I}$. If $\mathcal{G} \rightrightarrows M$ is source-connected, then $\{U_i\}_{i\in\mathtt{I}}$ is also an orbit cover for the underlying Lie algebroid $\mathbf{Lie}(\mathcal{G})$.

Theorem 5 *For an integrable Lie algebroid A with an orbit cover $\{U_i\}_{i\in I}$, let $\mathcal{G}_i \rightrightarrows U_i$ be a source-connected Lie groupoid and let $\phi_{ij} : (\mathcal{G}_i|_{U_{ij}})^c \to (\mathcal{G}_j|_{U_{ij}})^c$ be groupoid morphisms satisfying $\mathbf{Lie}(\phi_{ij}) = \mathrm{id}$, $\phi_{ii} = \mathrm{id}$, $\phi_{ij} = \phi_{ji}^{-1}$ and the cocycle condition. The fibered coproduct of manifolds*

$$\mathcal{G} = \coprod_{i\in I} \mathcal{G}_i \Big/ \sim$$

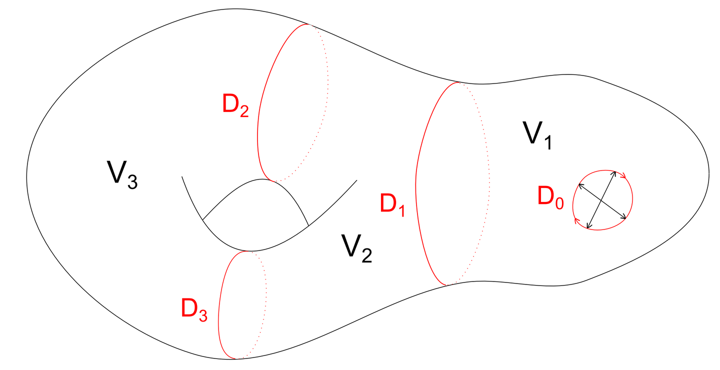
is a source-connected Lie groupoid integrating A, such that $(\mathcal{G}|_{U_i})^c = \mathcal{G}_i$. Moreover, every source-connected groupoid is obtained in this way.



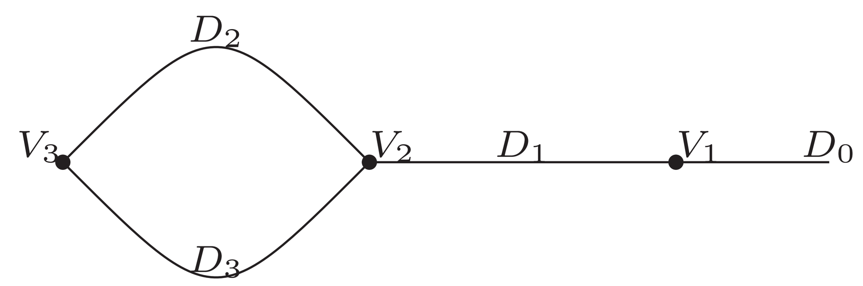
Theorem 5 enables us to classify the symplectic groupoids of a proper log symplectic manifold.

Theorem 6 *[1] For a proper log symplectic manifold (M,π), the symplectic groupoids are classified by a family of normal subgroups $K_i \lhd \pi_1(V_i, y_i)$ for each connected component $V_i \subset (M\backslash D)$ that 'agree' when pulling back to the symplectic leaves of the connected components of degeneracy locus D.*

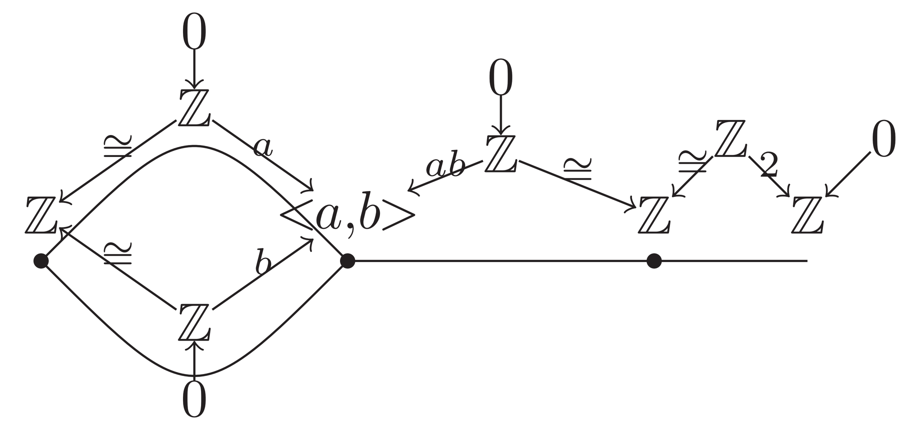
Example: log symplectic surface



For example, we associate a graph (below) with the log symplectic surface (above).



In addition, we label the vertices and (half-)edges with the fundamental groups of V_i, D_j and the symplectic leaf of D_j, and the kernel of the first Stiefel Whitney class of ND_j with the induced morphisms, as illustrated below.



The symplectic groupoids are classified by a family of normal subgroups for each of $\mathbb{Z}$, <a,b> and $\mathbb{Z}$.

In higher dimensions, Theorem 6 implies the source-simply-connected groupoid is Hausdorff if and only if, for each symplectic leaf F contained in D, and for each class $\gamma \in \pi_1(F)$ on which the first Stiefel-Whitney class of ND vanishes, the push-off of γ is nonzero in the fundamental group of the adjacent open symplectic leaf or pair of leaves.

Reference

[1] M. Gualtieri and S. Li, *Symplectic groupoids of log symplectic manifolds*, arXiv:1206.3674v1.

[2] V. Guillemin, E. Miranda and A. R. Pires, *Symplectic and Poisson geometry of b-manifolds*, arXiv:1206.2020v1.