1 Summary

My research interests are in differential geometry. Specifically, I am intereseted in Poisson geometry, and the mechanics and physics problems related to it. Poisson manifolds arise naturally as the phase space of a mechanical system. The study of Poisson manifolds will help us develop a better grasp of some of the most interesting problems in mathematics and physics, such as quantization, quantum field theory, dynamical systems and so on.

Like all other area of differential geometry, the central issue is to find the invariants of Poisson manifolds, which is computable in interesting examples. This leads us to the log symplectic manifolds, which is a class of even-dimensional Poisson manifolds that are generically symplectic.

Unlike the general Poisson manifolds where it is difficult to compute invariants, the log symplectic manifolds provide a non-trivial class of Poisson manifolds where a concrete understanding may be achieved. We provide several examples:

1. The Poisson cohomology which generalizes de Rham cohomology is often infinite-dimensional, but in collaborative work with Gualtieri, Pelayo and Ratiu [6], we manage to compute the finite-dimensional Poisson cohomology of a log symplectic manifold.

2. Although there is a criterion of when a Poisson manifold may be integrated to a symplectic groupoid [1, 2], we lack of a concrete understanding of the geometry of the symplectic groupoid. In collaborative work with Gualtieri [5], we manage to construct the symplectic groupoid of a log symplectic manifold by iterated blow-up's. This forms the basis of my thesis. [9]

3. The notion of momentum maps for general Poisson manifolds is not clear. However, in collaborative work with Gualtieri, Pelayo and Ratiu [6], we achieve a Delzant type classification for toric log symplectic manifolds.

4. As work in progress, we try to adapt the recent symplectic cohomology theory by Tsai-Tseng-Yau [15, 16, 14] to the log symplectic case.

As it often happens, the tools developed for one project have applications in others. For ODEs with irregular singularities, the solutions are often formal power series with zero radius of convergence. There are methods, by Borel, Ecalle, and Ramis, by which the formal solution may be "resummed" by to obtain actual solution. Reintepreting the problem in the language of groupoid representation, the blow-up construction of Lie groupoids yields a more direct approach to resummation [7].

The theory of Lie algebroids and Lie groupoids, important in Poisson geometry, also has implications in geometric mechanics. The classical variational principle on the tangent bundle TM yields the Euler-Lagrange equation. Generalizing it to a variational principle on $TM \oplus T^*M$, it simultaneously yields the A-path condition $v = \dot{q}$, the Legendre transform and the Euler-Lagrange equation [19]. In the other direction, the variational principle on a Lie algebroid A yields the Euler-Lagrange equation when $A = TM$ and the Euler-Poincare equation when $A = \mathfrak{g}$ is a Lie algebra [18, 11]. Combining the two ideas, we develop a variational principle on $A \oplus A^*$ that yields the A-path condition, the Legendre transform and the generalized Euler-Lagrange equation [10]. This variational principle on $A \oplus A^*$ may be interpreted as the reduction of a Lie groupoid action of the variational principle on a fibered manifold.

2 Past research

2.1 Poisson geometry

We begin with the physical relevance of Poisson geometry. In physics, the phase space M of a Hamiltonian mechanical system is equipped with a Lie bracket $\{\cdot,\cdot\}$, called the Poisson bracket, on the time-dependent functions $C^\infty(M\times\mathbb{R})$. The dynamics of the mechanical system is governed by the Hamilton's equation:

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\{f,\mathcal{H}\} \tag{2.1}$$

where $\mathcal{H}$, called the Hamiltonian function, is usually taken to be the energy. Mathematically, a Poisson manifold is a smooth manifold endowed with a Poisson bracket on the smooth functions. Equivalently, the Poisson bracket is characterized by a a bivector $\pi\in\Gamma(\bigwedge^2 TM)$ via the relation:

$$\{f,g\}=\pi(df\otimes dg),$$

where π satisfies $[\pi,\pi]=0$.

It turns out that the cotangent bundle T^*M of a Poisson manifold (M,π) is naturally a Lie algebroid. Thus, the study of Poisson geometry is intimately related to the theory of Lie algebroids and Lie groupoids.

We briefly recall the definitions of Lie groupoids and Lie algebroids. A Lie groupoid $G\rightrightarrows M$ is a manifold G over the base manifold M together with the structure maps

$$G_s\times_t G \xrightarrow{\ m\ } G \xleftarrow{\ \mathsf{id}\ } M, \qquad i: G\to G,\quad s,t: G\to M \tag{2.2}$$

where the source $s:G\to M$ and the target $t:G\to M$ are surjective submersions, and the identity $\mathsf{id}:M\to G$, the inverse $i:G\to G$ and the multiplication $m:G_s\times_t G\to G$ are smooth.

On the other hand, a Lie algebroid is a vector bundle A over the base manifold M together with Lie bracket on $\Gamma(A)$ and a bracket-preserving bundle map $\rho:A\to TM$, called the anchor map, such that the Leibniz rule is stastified. That is, for $X,Y\in\Gamma(A)$,

$$[X,fY]=f[X,Y]+\rho(X)(f)\cdot Y.$$

In the case when M is a point, we recover the usual notion of Lie groups and Lie algebras.

For a Poisson manifold (M,π), the cotangent bundle T^*M together with the Poisson anchor map

$$\pi^\flat: T^*M\to TM,\qquad \alpha\mapsto\iota_\alpha\pi \tag{2.3}$$

and the Koszul bracket:

$$[\alpha,\beta]=L_{\pi^\flat(\alpha)}\beta-L_{\pi^\flat(\beta)}\alpha-d\pi(\alpha,\beta),\qquad \alpha,\beta\in\Omega^1(M), \tag{2.4}$$

becomes a Lie algebroid. We denote this Lie algebroid by $T^*_\pi M$. The Poisson cohomology of (M,π), denoted by $H^\bullet_\pi(M)$, is the Lie algebroid cohomology of $T^*_\pi M$.

The Poisson structure also induces a multiplicative symplectic 2-form on the Lie groupoid integrating the Poisson algebroid $T^*_\pi M$'. By a multiplicative 2-form on $G\rightrightarrows M$, we mean a 2-form $\sigma\in\Omega^2(G)$ that satisfies

$$(p_1)^*(\sigma)-(p_2)^*(\sigma)=m^*(\sigma),$$

where $p_1:G_s\times_t G\to G$ and $p_2:G_s\times_t G\to G$ are the first and the second projections. A symplectic groupoid is then a Lie groupoid with a multiplicative symplectic 2-form [17]. We give a simple example.

When (M,π) is symplectic, i.e. the Poisson anchor (2.3) is an bundle isomorphism. Then the pair groupoid $M \times M \rightrightarrows M$ where $M \times M$ is equipped the symplectic structure $\pi \oplus -\pi$ is the symplectic groupoid of (M,π). Here, the source and target maps of the pair groupoid $M \times M \rightrightarrows M$ is the first and second projection respectively, and the identity map $\mathsf{id} : M \to M \times M$ is the diagonal embedding.

For a given Poisson manifold, the criterion of the existence of a symplectic groupoid integrating it was solved in [1, 2], but we lacked concrete examples of symplectic groupoids. For this reason, we introduce the notion of log symplectic manifolds.

Definition 2.1. Let M be a smooth manifold, and let L be the union of a finite number of normal-crossing hypersufaces, called a normal crossing divisor of M. The log tangent bundle $TM(-\log L)$ is the Lie algebroid with sheaf of sections

$$\Gamma\left(TM(-\log L)\right) = \{X \in \Gamma(TM) \mid X \text{ is tangent to } L\}.$$

A log symplectic form on (M,L) is then a non-degenerate closed log 2-form $\omega \in \Omega^2(M,-\log L)$.

The inverse $\omega^{-1} \in \Gamma(\bigwedge^2 TM(-\log L))$, of the log 2-form ω, is a Poisson structure, which is non-degenerate away from L. In collaborative work with Gualtieri, Pelayo and Ratiu [6], we computed the Poisson cohomology of a log symplectic manifold.

Theorem 2.2. *[6] Let (M,L,ω) be a log symplectic manifold. Let $\{L\}_{i\in I}$ be the smooth components of L, and let*

$$L^{(j)} = \bigcup_{i_1,\ldots,i_j \in I} L_{i_1} \cap \ldots L_{i_j}$$

be the j-fold of $\{L\}_{i\in I}$, i.e. $L^{(2)}$ is the double intersection, $L^{(3)}$ is the triple intersection and so on. We have the Poisson cohomology of (M,ω) as follows:

$$H^k_{\omega^{-1}}(M) = H^k(M) \oplus H^{k-1}(L) \oplus H^{k-2}(L^{(2)}) \oplus H^{k-3}(L^{(3)}) \oplus \ldots$$

When L is a smooth hypersurface, the local structure of a log symplectic form was studied by Guillemin et al. [8], where they used the terminology b-symplectic instead of log symplectic. The main result is that if L is compact and contains a compact symplectic leaf F, then L is a symplectic mapping torus, i.e.

$$L = F \ltimes_\varphi S^1 = \frac{F \times \mathbb{R}}{(x,t) \sim (\varphi(x), t+1)}$$

where $\varphi : F \to F$ is a symplectomorphism.

Now if we do a real projective blow-up of the pair groupoid $M \times M \rightrightarrows M$ along the subgroupoid $L \times_{S^1} L \rightrightarrows L$, and remove the proper transforms of the source preimage $s^{-1}(L)$ and the target preimage $t^{-1}(L)$, then the resulting space

$$[M \times M : L \times_{S^1} L] \subset \mathsf{Bl}_{L \times_{S^1} L}(M \times M)$$

is naturally a Lie groupoid over M. In fact, we proved that it is a symplectic groupoid of (M,ω).

Theorem 2.3. *[5] Let (M,L,ω) be a compact log symplectic manifold, where L is smooth, compact, connected and contains a compact symplectic leaf F. The Lie groupoid $[M \times M : L \times_{S^1} L] \rightrightarrows M$ equipped with the induced symplectic structure is a symplectic groupoid of (M,ω).*

In the same paper [5], we also gave another explicit gluing construction of the symplectic groupoid of a log symplectic manifold, and we manage to classify all possible symplectic groupoids of a given log symplectic manifold. In my thesis [9], we generalized it to the case when L is a normal crossing divisor, where we construct a symplectic groupoid by iteratively applying the same blow-up construction to each smooth component of L.

About two years ago, together with Gualtieri, Pelayo and Ratiu, we studied the toric log symplectic manifolds [6]. Unlike the toric symplectic manifold [4], where the momentum codomain is an affine space and the momentum image is a Delzant polytope, the momentum codomain of a toric log symplectic manifold is a log affine manifold, i.e. (M, D) together with a $\mathbb{R}^n$ action that trivializes $TM(-\log D)$. More precisely, we have the following result.

Theorem 2.4. *[4] There is a one-to-one correspondence between equivariant isomorphism classes of oriented compact connected toric Hamiltonian log symplectic $2n$–manifolds and equivalence classes of pairs (Δ, M), where Δ is a compact convex log affine polytope of dimension n satisfying the Delzant condition and M is a principal n–torus bundle over Δ with vanishing toric log obstruction class.*

Using this classification result, we also find some of the first examples of log symplectic manifolds that do not admit symplectic structures.

2.2 Ordinary differential equations with singularity

The blow-up construction of Lie groupoids is also interesting in the holomorphic category, where we replace the real projective blow-up with the complex projective blow-up. Let Σ be a Riemann surface, and let D be a divisor of Σ, i.e. D is a collection of points on Σ counted with multiplicity. By iteratively applying the blow-up construction to the pair groupoid $\Sigma \times \Sigma$ along the identity image of the points in D, we may construct the Lie groupoid $\mathrm{Pair}(X, D)$ integrating the holomorphic log tangent algebroid $T\Sigma(-\log D)$.

Interpreting a meromorphic connection ∇ with poles on D as the Lie algebroid representations of $T\Sigma(-\log D)$, we may integrate these meromorphic connections to be the Lie groupoid representation of $\mathrm{Pair}(X, D)$. Because ∇ has the irregular singularity, the solutions of $\nabla s = 0$ are formal power series of around the singular points. By pulling s back to the groupoid, we obtain a converging power series on the groupoid, which is nothing but a groupoid representation of $\mathrm{Pair}(X, D)$. We may then obtain actual solutions by applying this representation to a given initial condition. This is collaborative work with Gualtieri and Pym. [7]

2.3 Geometric mechanics

Most recently, in collaboration with Xiang Tang and Ari Stern, we have finished a project in geometric mechanics. We extended the Hamilton-Pontryagin principle by Yoshimura and Marsden [19], which is a variational principle on $TQ \oplus T^*Q$, to a variational principle of $A \oplus A^*$. Beginning with a usual Lagrangian function $L : C^\infty(TQ)$, Yoshimura and Marsden consider its pull back to $TQ \oplus T^*Q$ and the space of path on $TQ \oplus T^*Q$. The resulting variational principle yields the implicit Euler-Lagrange equations, which consists of

1. A-path condition: $v = \dot{q}$,
2. the Legendre transform: $p = \frac{\partial L}{\partial v}$,
3. and the Euler-Lagrange equation: $\dot{p} = \frac{\partial L}{\partial q}$.

When Q is a Lie group G, the reduction of the Hamilton-Pontryagin principle for a right-invariant Lagrangian L on TG yields the implicit Euler-Poincare equations on $\mathfrak{g} \oplus \mathfrak{g}^*$: [20]

$$\xi = \eta, \quad \mu = \frac{\delta l}{\delta \eta}, \quad \dot{\mu} = \mathtt{ad}^*_{\xi}\mu. \tag{2.5}$$

For a surjective submersion $\mu : Q \to M$, let L be a Lagrangian function on the vertical bundle VQ. We consider its pull-back to $VQ \oplus V^*Q$ and the space of vertical paths. The resulting variational principle, which we also called the Hamilton-Pontryagin principle, yields a similar set of equations as in the implicit Euler-Lagrange equations above.

However when a Lie groupoid $G \rightrightarrows M$ acts freely and properly on Q, the reduction of the Hamilton-Pontryagin principle for a G-invariant Lagrangian L on VG is more involved. To begin with, the reduced variational principle is on $\mathfrak{at} \oplus \mathfrak{at}^*$, where the Lie algebroid $\mathfrak{at} = VQ/G$ is the Atiyah algebroid. The kind of pathes we consider consists

1. $a \in \mathcal{P}_\rho(A)$ is an A-path on A with the base path $\sigma \in \mathcal{P}(M)$, i.e. $\frac{d\sigma}{dt} = \rho \circ a$;
2. $v \in \mathcal{P}(A)$ is a path on A covering σ;
3. $p \in \mathcal{P}(A^*)$ is a path on A^* covering σ.

The resulting variation principle yields what we called the implicit Euler-Lagrange-Poincare equations:

1. $a = v$, which is equivalent to the A-path condition $\rho \circ v = \frac{d\sigma}{dt}$,
2. the Legendre transform: $p = \frac{\partial l}{\partial v}$,
3. and the Euler-Lagrange-Poincare equation: $\rho^* \frac{\partial l}{\partial \sigma} + \overline{\nabla}^*_a(p) = 0$.

As an application, the reduction theory gives a more direct approach to Routhian reduction, which is the Lagrangian counterpart of the symplectic reduction in Hamiltonian mechanics.

3 Current projects and future plans

There are many possible directions to continue of my work on log symplectic manifolds. The Morita equivalence, which is a coarse equivalence on Poisson manifolds, is understood for log symplectic manifolds in dimension 2 [13]. I have worked out a sufficient condition for Morita equivalence of general log symplectic manifolds, but it is not clear if it is necessary. Furthuremore, even for a 2-dimensional log symplectic manifold, which has been completely classified [12], the symplectic topology of its symplectic groupoid is not known. In colloboration with Gualtieri, we use the idea of Poisson spray [3] to tackle the problem.

In pursuit of finding more invariants for log symplectic manifolds, I recently began exploring the symplectic cohomology theory by Tsai-Tseng-Yau [15, 16, 14]. It is very likely that we will be able to prove the analogue theory for log symplectic manifold yields a finite-dimensional cohomology theory, and hopefully we will find interesting examples in which this new invariant will provide new insights.

It would be also interesting to explore the Seiberg-Witten invariants and the geometric quantization scheme in the setting of log symplectic manifolds.

Together with Xiang Tang, we are also exploring a possible connection between the Hamilton-Pontryagin principle to the AKSZ-BV formalism.

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