Toric log symplectic manifolds

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Outline

Toric symplectic manifolds

A **toric symplectic manifold** is a triple (M, ω, T^n) such that

- M is a compact 2n-manifold;
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- there is an effective action on (M, ω) by $T^n = (S^1)^n$.

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We say (M, ω, T^n) is **Lagrangian**, if for $a, b \in \mathfrak{t}$,

$$\omega(\rho(a), \rho(b)) = 0.$$

Hamiltonian action

We say (M, ω, T^n) is **Hamiltonian**, if there exists a momentum map $\mu : M \to \mathfrak{t}^*$ such that for every $a \in \mathfrak{t}$,

$$\iota_{\rho(a)}\omega = d\mu_a.$$

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If we write the momentum image as $\Delta = \mu(M)$, then equivalently, (M, ω, T^n) is Hamiltonian if $\iota_\rho\omega$ descends to $\xi \in \Omega^1(\Delta) \otimes \mathfrak{t}^*$ which satisfies

$$[\xi] = 0 \in H^1(\Delta) \otimes \mathfrak{t}^*.$$

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(M, ω, T^n) is Hamiltonian $\implies$ (M, ω, T^n) is Lagrangian.

Delzant polytopes

If we treat $\mathfrak{t}^*$ as an affine space, then its (co)tangent bunlde is trivilized.

$$T\mathfrak{t}^* = \mathfrak{t}^* \times \mathfrak{t}^*, \qquad T^*\mathfrak{t}^* = \mathfrak{t}^* \times \mathfrak{t}$$

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A Delzant polytope is convex polytope $\Delta \subset \mathfrak{t}^*$ is a convex polytope such that

- A face f $\subset \Delta$ has a rational normal vector v_f;
- At each vertex v, the vectors v_f of the faces meeting v spans $\mathfrak{t}^*_\mathbb{Z}$.

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Delzant Theorem

There is a one-to-one bijection between Hamiltonian toric symplectic manifolds (M, ω, T^n, μ) and the Delzant polytopes Δ.

$\Delta \rightsquigarrow (M, \omega, T^n)$

Given a Delzant polytope $\Delta \subset \mathfrak{t}^*$, the natural symplectic structure on the cotangent bundle

$$T^*\Delta = \Delta \times \mathfrak{t}$$

is invariant under the $\mathfrak{t}_{\mathbb{Z}}$ translation on the fibers. We denote the quotient by

$$(\widetilde{M} = T^*\Delta/\mathfrak{t}_{\mathbb{Z}}, \widetilde{\omega})$$

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By performing the symplectic cuts along the boundary components of $(\widetilde{M}, \widetilde{\omega}, T^n)$, we obtain a compact Hamiltonian toric symplectic manifold (M, ω, T^n).

Toric log symplectic manifolds

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- M is a compact 2n-manifold;
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The notion of Hamiltonian is a bit tricky, so we need define the notion of log affine manifolds.

Log affine manifolds

A **log affine manifold** is a triple (Δ, D, ξ) such that

- Δ is a n-manifold with corners;
- $D = \bigcup D_i$ is a collection of normal crossing hypersurfaces with $\partial\Delta \subset D$;
- $\xi \in \Omega^1(X, \log D) \otimes \mathfrak{t}^*$ is a $\mathfrak{t}^*$-valued log 1-form trivializing the log tangent bundle $T(X, -\log D)$.

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Proposition (Gualtieri-L-Pelayo-Ratiu)

If (M, Z, ω, T^n) is Lagrangian toric, then

$$(\Delta = M/T^n, D = Z/T^n, \xi = \iota_\rho \omega)$$

is log affine. ($\iota_\rho \omega$ is basic and descends ξ.)

Hamiltonian action

The logarithmic cohomology of (Δ, D) decomposes as follows:

$$H^n(M, \log Z) = H^n(M) \oplus \left(\bigoplus_i H^{n-1}(Z_i) \right) \oplus \left(\bigoplus_{i>j} H^{n-2}(Z_i \cap Z_j) \right) \oplus \dots$$

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A Lagrangian toric (M, Z, ω, T^n) is **Hamiltonian** if

$$\iota_\rho \omega \in \Omega^1(M, \log Z) \otimes \mathfrak{t}^*$$

has trivial class in $H^1(M) \otimes \mathfrak{t}^*$. This is equivalent to say that $\iota_\rho \omega$ descends to

$$\xi \in \Omega^1(\Delta, \log D) \otimes \mathfrak{t}^*$$

which has trivial class in $H^1(\Delta) \otimes \mathfrak{t}^*$, i.e. (Δ, D, ξ) has trivial **affine monodromy**.

Goal: Classify the Hamiltonian (M, Z, ω, T^n) using (Δ, D, ξ).

Tropical domains

Idea: Think $\mathfrak{t}^*$ as an affine space, and compatifiy $\mathfrak{t}^*$ to a log affine space, which is called a tropical domain.

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Example 1. Compactifying $\mathbb{R}$ at $\pm\infty$, we obtain a tropical domain isomorphic to the closed interval $[0, 1]$ with a log 1-form

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Example 2a. Compactifying $\mathbb{R}^2$ in the (reserve) directions of

$$(1, 0), (0, 1), (-1, -1)$$

we obtain a tropical domain isomorphic to the triangle

$$X = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y, \geq 0, x + y \leq 1\}$$

with the log 1-form $\xi \in \Omega^1(X, \partial X) \otimes \mathbb{R}^2$ such that the residues of ξ are exactly $(1, 0)$, $(0, 1)$, and $(-1, -1)$.

Tropical welding

In general, we may pick any fan Σ of $\mathfrak{t}^*$ and partially compactify $\mathfrak{t}^*$ in the (reverse) directions of the vectos generating its 1-dim cones to obtain an **tropical domain** X_Σ.

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For two such tropical domains (X, ξ) and (X', ξ'), we may weld X and X' along the boundary components D_i and D'_j if

$$\mathrm{Res}_i \xi = \mathrm{Res}_j \xi' \in \mathfrak{t}^*$$

and all adjacent residues also match. The result of such welding is again a log affine manifold, which is called a **tropical welded space**.

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Example 2b. We may glue 8 copies of the triangles in Example 2a to obtain a log affine manifold isomorphic to S^2 with $\xi \in \Omega^1(S^2, \log D) \otimes \mathbb{R}^2$ where D is 3 transversally intersecting great circles.

Log affine polytopes

Let (X, D, ξ) be a tropical welded space. We say $\Delta \subset X$ is a **Dezlant log affine polytope** if Δ is compact and intersets the interior of each tropical domain in a (possibly non-compact) Delzant polytope.

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If (M, Z, ω, T^n) is Hamiltonian toric, then

$$(\Delta = M/T^n, D = Z/T^n, \xi = \iota_\rho \omega)$$

is a Delzant log affine polytope.

Chern class

Let (M, Z, ω, ξ) be a Lagrangian toric log symplectic manifold. In particular, the T^n action is locally free. There is a reverse operation to symplectic cut, which we called symplectic uncut, which renders the T^n action free.

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Let M' be the resulting uncut manifold. Then M' is a T^n principal bundle over Δ. The n-tuple Chern class of (M, Z, ω, T^n) is

$$c_1(M) = c_1(M') = (c_1^1, \ldots, c_1^n) \in H^2(\Delta) \otimes \mathfrak{t}$$

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Lemma (Gualtieri-L-Pelayo-Ratiu)

For a Lagrangian toric log symplectic manifold (M, Z, ω, ξ),

$$c_1(M) \cdot \xi = \sum_{j=1}^{n} c_1^j \wedge [\xi_j] = 0 \in H^3(\Delta, \log D).$$

Delzant correspondence

Thereom (Gualtieri-L-Pelayo-Ratiu)

The classification of Hamiltonian log symplectic manifolds are as follows:

$$
\begin{aligned}
& (M, Z, \omega, T^n) \text{ Hamiltonian log symplectic manifold} \\
\Longleftrightarrow \quad & (\Delta, D, \xi) \text{ Delzant log affine polytope} \\
& + c_1 \in \Omega^2(\Delta) \otimes \mathfrak{t} \text{ such that } c_1(M) \cdot \xi = 0 \\
& + \text{ the affine space } H^2(\Delta, \log D)
\end{aligned}
$$

Examples

Example 2c. In Example 2c, we have a log affine manifold (S^2, D, ξ) where D is 3 transversally intersecting great circles. If we take $c_1 = 0$, then $c_1 \cdot \xi = 0$ holds. This also implies

$$M = S^2 \times T^2, \qquad Z = D \times T^2.$$

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Since $T(T^2) = T^2 \times \mathfrak{t}$, we have a natural $\mathfrak{t}$-valued 1-form $\theta = (\theta^1, \ldots, \theta^n) \in \Omega^1(T^2) \otimes \mathfrak{t}$. For any $\beta \in H^2(S^2, \log D)$, we take

$$\omega = \sum_{j=1}^{n} \theta^j \wedge \xi_j + \beta \in \Omega^2(M, \log Z).$$

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$$\omega = \sum_{j=1}^{n} \theta^j \wedge \xi_j + \beta \in \Omega^2(M, \log Z).$$

Then (M, Z, ω, T^2) is a Hamiltonian toric log symplectic manifold.
Note: $H^2(S^2, \log D) = \mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^6 = \mathbb{R}^{10}$ and we have a 10-dim moduli space of Hamiltonian toric log symplectic structures on (M, Z, T^2).

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Example 2c. cont... Another way to think about Example 2c is as follows. Because ξ trivializes $T(S^2, -\log D)$, so ξ also trivializes $T^*(S^2, \log D)$, i.e.

$$T^*(S^2, \log D) \cong S^2 \times \mathfrak{t}$$

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Let π_0 be the natural Poisson structure on $T^*(S^2, -\log D)$. Then

$$M = T^*(S^2, \log D)/\mathfrak{t}_{\mathbb{Z}}$$

and π_0 descends to a Poisson structure on M whose inverse is

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Remark: There is no distinguished ω_0 if the chern class $c_1 \neq 0$.

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Half of the surface, i.e. a torus with a disk removed, is a log affine polytope. Taking the Chern class $c_1 = 0$, we obtain a 4-dim Hamiltonian log symplectic manifold (M, Z, ω, T^2) such that

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Interestingly, M has vanishing Sieberg-Witten invariant, so M is not symplectic, but M is log symplectic.

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Thank you!