Mechanics on fibered manifolds

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Outline

Lagrangian mechanics

- Lagrangian mechanics on TQ
- Lagrangian mechanics on A
- Reduction by groupoid action

Hamilton-Pontryagin mechanics

- H-P mechanics on $TQ \oplus T^*Q$
- H-P mechanics on (A, A^*)
- Reduction by groupoid action

- Q is a manifold.
- $L \in C^\infty(TQ)$ is a smooth function.
- $\mathcal{P}(Q) = \{q : I \to Q \mid q \text{ is } C^2\}$.
- $\delta q \in T_q\mathcal{P}(Q)$ with $\delta q(0) = \delta q(1) = 0$.
- $S : \mathcal{P}(Q) \to \mathbb{R}, \qquad S(q) = \int_0^1 L(q(t), \dot{q}(t))dt$.

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Lagrangian mechanics on TQ

A path $q \in \mathcal{P}(Q)$ satisfies Hamilton's variational principle if

$$\delta S = dS(\delta q) = 0, \quad \forall\, \delta q. \tag{1}$$

In coordinates, we have the Euler-Lagrange equation:

$$\frac{\partial L}{\partial q^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}. \tag{2}$$

- Q is a manifold
- $\rho : A \to TQ$ is a Lie algebroid.
- $\mathcal{P}_\rho(A) = \{a : I \to A \mid \rho(a) = \dot{q}\}$ is the space of A-paths, where q is the base path of a.

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Choosing TQ-connection on A, $\nabla : \Gamma(TQ) \times \Gamma(A) \to \Gamma(A)$, we have two induced A-connections on A, ∇ and $\overline{\nabla}$:

$$\nabla_X Y = \nabla_{\rho(X)} Y, \qquad \overline{\nabla}_X Y = \nabla_{\rho(Y)} X + [X, Y]. \tag{3}$$

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An A-variation of a is $\delta a = X_{b,a} \in T_a\mathcal{P}_\rho(A)$ where $b \in \mathcal{P}(A)$ is a path in A such that $b(0) = 0$ and $b(1) = 0$.
Reletive to a chosen TM-connection ∇ on A, the horizontal component of $X_{b,a}$ is $\rho(b)$, and the vertical component is $\overline{\nabla}_a b$, which are independent of the choice of ∇.

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Lagrangian mechanics on A (Weinstein '96, Martinez '01, L-S-T)

A A-path $a \in \mathcal{P}_\rho(A)$ satisfies Hamilton's variational principle if

$$\delta S = dS(X_{b,a}) = 0, \quad \text{for all } A\text{-variations } X_{b,a}. \tag{4}$$

Using ∇, we have the Euler-Lagrange-Poincare equation:

$$\rho^* \mathrm{d}L^{\mathrm{hor}}(a) + \overline{\nabla}_a^* \mathrm{d}L^{\mathrm{ver}}(a) = 0. \tag{5}$$

- $G \rightrightarrows M$ is a Lie groupoid.
- $\mu : Q \to M$ is a fibered manifold equipped with a free and proper action by $G \rightrightarrows M$.
- VQ is the vertical tangent bundle.
- $L \in C^\infty(VQ)^G$ is G-invariant.

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The mechanics for L on VQ is equivalent to the mechanics for ℓ on VQ/G. That is,

$$\begin{aligned} &\widetilde{a} \in \mathcal{P}_V(Q) \text{ satisfies the E-L-P equation for } L \\ \iff\ & a \in \mathcal{P}_\rho(VQ/G) \text{ satisfies the E-L-P equation for } \ell. \end{aligned}$$

Example

- G is a Lie group.
- $L \in C^\infty(TQ)^G$ is (left) invariant.

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- $L \in C^\infty(TG)^G$ is (left) invariant.

It follows that L reduces to $\ell \in C^\infty(\mathfrak{g})$. The Euler-Lagrange equation for L reduces to the Euler-Poincare equation:

$$\mathrm{ad}^*_\xi \mu = \dot{\mu}, \tag{6}$$

where $\xi(t) = a(t)$ and $\mu = \frac{\delta \ell}{\delta \xi}$.

- Q is a manifold.
- $L \in C^\infty(TQ)$ is a smooth function.
- $\mathcal{P}(TQ \oplus T^*Q) = \{(q, v, p) : I \to TQ \oplus T^*Q\}$.
- $(\delta q, \delta v, \delta p) \in T_{q,v,p}\mathcal{P}(TQ \oplus T^*Q)$ with $\delta q(0) = \delta q(1) = 0$.
- $S : \mathcal{P}(TQ \oplus T^*Q) \to \mathbb{R}$ is defined by

$$S(q, v, p) = \int_0^1 L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t)\rangle \, dt. \tag{7}$$

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H-P mechanics on $TQ \oplus T^*Q$

A path $(q, v, p) \in \mathcal{P}(TQ \oplus T^*Q)$ satisfies H-P principle if

$$\delta S = dS(\delta q, \delta v, \delta p) = 0, \quad \forall\, (\delta q, \delta v, \delta p). \tag{8}$$

In coordinates, we have the implicit E-L equations:

$$\dot{p}_i = \frac{\partial L}{\partial q^i}, \qquad p_i = \frac{\partial L}{\partial v^i}, \qquad v_i = \dot{q}^i. \tag{9}$$

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(A, A^*)-paths

An (A, A^*)-path consists of

1. an A-path $a \in \mathcal{P}_\rho(A)$ over the base path $q \in \mathcal{P}(Q)$;
2. a path on A, $v \in \mathcal{P}(A)$, over q;
3. a path on A^*, $p \in \mathcal{P}(A^*)$, over q;

The space of (A, A^*)-paths: $\mathcal{P}(A, A^*)$.

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The action S

$S : \mathcal{P}(A, A^*) \to \mathbb{R}$ is defined by

$$S(a, v, p) = \int_0^1 L\left(v(t)\right) + \langle p(t), a(t) - v(t)\rangle \, dt. \tag{10}$$

Variations of (A, A^*)-paths

For a (A, A^*)-path (a, v, p),

1. we vary the A-path a by an A-variation $\delta a = X_{b,a} \in T_a\mathcal{P}_\rho(A)$ with $b(0) = 0$ and $b(1) = 0$, whose base variation is $\delta q \in T_q\mathcal{P}(Q)$;
2. we vary the $v \in \mathcal{P}(A)$ by a free variation $\delta v \in T_v\mathcal{P}(A)$ over δq;
3. we vary the $p \in \mathcal{P}(A^*)$ by a free variation $\delta p \in T_p\mathcal{P}(A^*)$ over δq;

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3. we vary the $p \in \mathcal{P}(A^*)$ by a free variation $\delta p \in T_p\mathcal{P}(A^*)$ over δq;

A (A, A^*)-path (a, v, p) satisfies H-P principle if

$$\delta S = dS(\delta a, \delta v, \delta p) = 0, \quad \text{for all variations } (\delta a = X_{b,a}, \delta v, \delta p). \tag{11}$$

Theorem (Li-Stern-Tang)

Choosing a *TM*-connection ∇ on A, an (A, A^*)-path (a, v, p) satisfies H-P principle if and only if (a, v, p) satisfies the implicit E-L-P equations:

$$\begin{aligned} &\rho^* \mathrm{d}L^{\mathrm{hor}}(v) + \overline{\nabla}^*_a p = 0, \\ &\mathrm{d}L^{\mathrm{ver}}(v) - p = 0, \\ &a = v. \end{aligned} \tag{12}$$

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Theorem (Li-Stern-Tang)

The H-P mechanics for L on $VQ \oplus V^*Q$ is equivalent to the H-P mechanics for ℓ on (A, A^*) where $A = VQ/G$. That is,

$(\widetilde{q}, \widetilde{v}, \widetilde{p}) \in \mathcal{P}_V(VQ \oplus V^*Q)$ satisfies the implicit E-L-P equation for L.

$\iff$ $(a, v, p) \in \mathcal{P}(A, A^*)$ satisfies the implicit E-L-P equation for ℓ.

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Thank you!