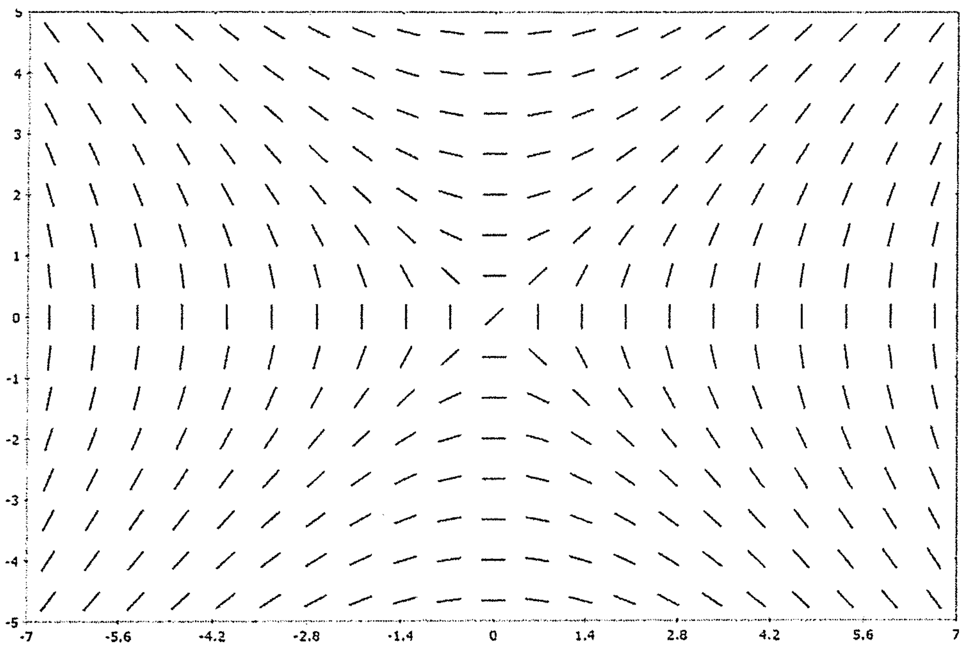
Math 217 Exam 1 Sept 17, 2015

Instructions:

1. There are three parts in this exam. Part I is multiple choice, Part II is True/False, and Part III consists of hand-graded problems.
2. The total number of points is 100.
3. You may use a calculator.
4. The scorecard and Part III will be collected at the end of the exam. You may take Part I and Part II with you at the end of the exam.

Part I. Multiple Choices $\quad 5 \times 10 = 50$ points

1. The direction field below corresponds which differential equation?



A. $y' = \frac{x}{y}$
B. $y' = x^2 + y^2$
C. $y' = -\frac{x}{y}$
D. $y' = x + y$
E. $y' = x - y$
F. $y' = x^2 - y^2$

A

2. Classify the differential equation $(\sin t)y'' + ty^2 = e^t$.

A. ordinary, linear, order 1

B. ordinary, linear, order 2

C. ordinary, non-linear, order 1

D. ordinary, non-linear, order 2

E. partial, linear, order 1

F. partial, non-linear, order 2

D

3. Which of the following is a solution of the differential equation $y' = y + 2e^{-t}$?

A. $y(t) = e^{-t}$
B. $y(t) = \sin t$
C. $y(t) = e^t + e^{-t}$
D. $y(t) = e^t - e^{-t}$
E. $y(t) = \cos t$
F. none of the above

D If $y = e^t - e^{-t}$,

then $y' = e^t + e^{-t}$

$y + 2e^{-t} = e^t - e^{-t} + 2e^{-t} = e^t + e^{-t}$

so $y' = y + 2e^{-t}$

4. The differential equation

$$\frac{xy'}{y} + e^y y' + \cos x + \ln y = 0$$

belongs to only one of the following categories. Which one is it?

A. 1st order linear

B. separable

C. exact

D. 2nd order linear

E. 2nd order non-linear

F. autonomous

C

$$\left(\frac{x}{y} + e^y\right) y' + (\cos x + \ln y) = 0$$

$$\frac{\partial}{\partial x}\left(\frac{x}{y} + e^y\right) = \frac{1}{y}$$

$$\frac{\partial}{\partial y}(\cos x + \ln y) = \frac{1}{y}$$

so. it is an exact equation.

5. Solve the initial value problem

$$y' + 3t^2 y = 0, \quad y(0) = 7.$$

A. $y(t) = e^{-t}$
B. $y(t) = 7e^{-t}$
C. $y(t) = \frac{7}{2}(e^{-t^2} + e^{t^2})$
D. $y(t) = 7e^{3t^2}$
E. $y(t) = 7e^{-t^3}$
F. none of the above

E

$$\mu(t) = e^{\int 3t^2 dt} = e^{t^3}$$

$$\frac{d}{dt}(e^{t^3} y) = 0$$

$$e^{t^3} y = C.$$

$$y = C e^{-t^3}$$

$y(0) = 7$ implies $C = 7$.

so $y = 7e^{-t^3}$

You may also solve it as a separable equations.

6. If ϕ is a solution of $y' = 6t(y-1)^{\frac{2}{3}}$ and $\phi(0) = 1$, what is $\phi(1)$?

A. 0
B. 1
C. 2
D. 9
E. 33
F. none of the above

Both **B** and **C** are correct, so you will get credit for this problem no matter what you choose.

$$\frac{dy}{dt} = 6t(y-1)^{\frac{2}{3}}$$

Case 1: $y = 1$ is a solution. so B is correct

Case 2: $(y-1)^{-\frac{2}{3}}\,dy = 6t\,dt$

$$3(y-1)^{\frac{1}{3}} = 3t^2 + C$$

$$\phi(0) = 1 \Rightarrow C = 0.$$

$$3(\phi(1)-1)^{\frac{1}{3}} = 3 \times 1^2 = 3.$$

$$\phi(1) = 2.$$

so C is correct.

7. Determine the interval in whilh the solution exists for the initial value problem

$$y' + y^3 = 0, \quad y(0) = 1.$$

A. $(-\infty, -\frac{1}{2})$
B. $(-\frac{1}{2}, \infty)$
C. $(-\infty, 0)$
D. $(0, \infty)$
E. $(-\frac{1}{2}, \frac{1}{2})$
F. none of the above

B

$$\frac{dy}{dt} = -y^3$$

$$y^{-3}\, dy = -dt$$

$$-\frac{1}{2} y^{-2} = -t + C$$

$$y(0) = 1 \implies C = -\frac{1}{2}.$$

$$-\frac{1}{2} y^{-2} = -t - \frac{1}{2}.$$

$$y^2 = \frac{1}{2t+1}.$$

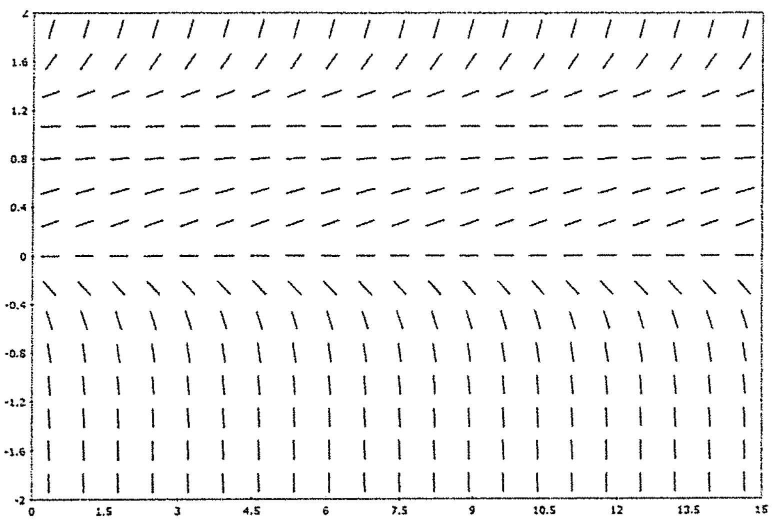
$$y = \pm\sqrt{\frac{1}{2t+1}}.$$

Using $y(0)=1$, we find $y(t) = \frac{1}{\sqrt{2t+1}}$.

Clearly, $2t+1 > 0$.

$t > -\frac{1}{2}$.

8. Only one of the statements is true for the direction field below. Which one is it?



A. $y(t) = 0$ is a stable equilibrium solution.

B. $y(t) = 0$ is an unstable equilibrium solution.

C. $y(t) = 1$ is a stable equilibrium solution.

D. $y(t) = 1$ is an unstable equilibrium solution.

E. Neither $y(t) = 0$ nor $y(t) = 1$ are equilibrium solutions.

F. None of the above.

B

9. Find an integrating factor μ that will make the following equation exact.

$$y + 2tyy' = e^{-2y}y'$$

A. $\mu = e^{2y - \ln y}$

B. $\mu = \frac{1-2y}{2ty - e^{-2y}}$

C. $\mu = e^{2t - \ln t}$

D. $\mu = e^{\frac{1}{2t}}$

E. $\mu = e^{-\frac{1}{2y}}$

F. None of the above.

A

$$(2ty - e^{-2y})y' + y = 0$$

$$M = y, \quad N = 2ty - e^{-2y}$$

$$\frac{\frac{\partial N}{\partial t} - \frac{\partial M}{\partial y}}{M} = \frac{2y-1}{y} = 2 - \frac{1}{y}.$$

$$\mu(y) = e^{\int 2 - \frac{1}{y}\,dy} = e^{2y - \ln y}$$

10. Which of the initial value problem below DOES NOT have a unique solution?

A. $y' = 0, \qquad y(0) = 0$
B. $y' = y^{\frac{1}{2}}, \qquad y(0) = 0$
C. $y' = y, \qquad y(0) = 0$
D. $y' = y^2, \qquad y(0) = 0$
E. $y' = y^{2015}, \qquad y(0) = 0$
F. None of the above.

B

$\frac{d(y^{\frac{1}{2}})}{dy} = \frac{1}{2} y^{-\frac{1}{2}}$ is not continuous at 0.

Part II. True/False $5 \times 2 = 10$ points

Choose 'A' if the statement is true; choose 'B' if the statement is false.

11. Only 1st order differential equations can be solved.

12. The function $y(t) = -2t$ is an equilibrium solution to the differential equation $y' = y + 2t$.

13. Consider the autonomous equation $\frac{dy}{dt} = f(y)$, where f is a continuous function. It is NOT possible to have two stable equilibrium solutions with no other equilibrium solution between them.

14. The function $\mu(t) = e^{2t}$ is an integrating factor for the equation

$$ty' + 2y + e^t = 0.$$

15. There is only one solution to the initial value problem:

$$y' = 5t^2, \qquad y(1) = 2\pi.$$

B B A B A

12. $y = -2t$ is NOT a solution

13. Draw a picture!
Alternatively, the statement can be proven using the intermediate value theorem.

14. $$y' + \frac{2}{t} y + \frac{e^t}{t} = 0.$$

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln t + C}.$$

for example, $\mu = e^{2\ln t} = t^2$ is an integrating factor.

15. yes, by Uniqueness and Existence theorems.

Part III. Hand-graded problems $\qquad 10 + 10 + 20 = 40$ points

16. (10 points)
Solve the initial value problem below.

$$t\frac{dy}{dt} + 2y = \sin t, \qquad y\left(\frac{\pi}{2}\right) = 0.$$

$$y' + \frac{2}{t}y = \frac{\sin t}{t}$$

$$\mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln t + B}$$

say $\mu(t) = \cancel{}\, e^{2\ln t} = t^2.$

$$\frac{d}{dt}(t^2 y) = t\sin t$$

$$t^2 y = \int t\sin t\,dt$$

$$= \int t(-\cos t)'\,dt$$

$$= t(-\cos t) - \int(-\cos t)\,dt$$

$$= -t\cos t + \sin t + C$$

$$y(t) = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{C}{t^2}.$$

$$y\left(\frac{\pi}{2}\right) = 0 \Rightarrow 0 = 0 + \frac{1}{(\frac{\pi}{2})^2} + \frac{C}{(\frac{\pi}{2})^2} \Rightarrow C = -1.$$

$$y(t) = -\frac{\cos t}{t} + \cancel{\sin t}\ \frac{\sin t}{t^2} - \frac{1}{t^2}$$

17. (10 points)
Solve the differential equation below.

$$\frac{dy}{dx} = \frac{x^2 - y^2}{x^2 - xy}$$

Hint: Try the substitution $v = \frac{y}{x}$.

$$v = \frac{y}{x}$$

$$\Rightarrow y = xv \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

$$\frac{dy}{dx} = \frac{x^2-y^2}{x^2-xy} = \frac{1-\left(\frac{y}{x}\right)^2}{1-\frac{y}{x}}.$$

$$v + x\frac{dv}{dx} = \frac{1-v^2}{1-v}$$

$$x\frac{dv}{dx} = \frac{1-v^2}{1-v} - v = 1+v-v = 1.$$

$$dv = \frac{dx}{x}.$$

$$v = \ln x + c$$

$$\frac{y}{x} = \ln x + c$$

$$y = x\ln x + cx.$$

16. (10 + 10 = 20 points)

The radiation of a black body is governed by the the Stefan-Boltzmann law. From it, we could derive a model for the variation of the temperature of a body with respect to its surroundings. The model is described by the differential equation

$$\frac{du}{dt} = -\alpha \left(u^4 - T^4\right)$$

where $u(t)$ is the temperature of the body at time t measured in Kelvin, T is the ambient temperature which we keep constant, and α is an constant.

(a) Solve the differential equation.

Note: An implicit solution is good enough. You need to do a rather difficult integration.

$$\frac{1}{u^4 - T^4}\, du = -\alpha\, dt$$

$$\frac{1}{(u^2+T^2)(u^2-T^2)}\, du = -\alpha\, dt$$

partial fraction:

$$\frac{1}{2T^2}\left(-\frac{1}{u^2+T^2} + \frac{1}{u^2-T^2}\right) du = -\alpha\, dt$$

partial fraction again:

$$-\frac{1}{2T^2}\frac{1}{u^2+T^2} + \frac{1}{2T^2}\cdot\frac{1}{2T}\left(-\frac{1}{u+T} + \frac{1}{u-T}\right) du = -\alpha\, dt$$

Integrate both sides:

$$-\frac{1}{2T^3}\arctan\left(\frac{u}{T}\right) - \frac{1}{4T^3}\cdot\ln\left(\frac{u}{T}+1\right) + \frac{1}{4T^3}\ln\left(\frac{u}{T}-1\right) = -\alpha t + C$$

$$-\frac{1}{2T^3}\arctan\left(\frac{u}{T}\right) + \frac{1}{4T^3}\ln\left(\frac{u-T}{u+T}\right) = -\alpha t + C.$$

(b) Now assume that the object is in vacumm, i.e. $T = 0K$. Then the model is simplified to be

$$\frac{du}{dt} = -\alpha u^4$$

Let us say $\alpha = 2.0 \times 10^{-10} K^{-3}/s$, the initial temperature $u(0) = 300K$. Calculate the time for the temperature of the body to reach $150K$. (Round it to the nearest second.)

$$\frac{du}{dt} = -\alpha u^4$$

$$\frac{du}{u^4} = -\alpha\, dt$$

$$-\frac{1}{3u^3} = -\alpha t + C$$

$$u(0) = 300 \Rightarrow C = -\frac{1}{3 \times 300^3}.$$

$$-\frac{1}{3 \times 150^3} = -2 \times 10^{-10}\, \tau - \frac{1}{3 \times 300^3}$$

$$\tau \cong 432 \text{ sec}$$

So it takes 7 min 12 sec to cool down to 150K.