Math 217 Exam 2 Solution   Oct 20, 2015

Instructions:

1. There are three parts in this exam. Part I is multiple choice, Part II is True/False, and Part III consists of hand-graded problems.
2. The total number of points is 100.
3. You may use a calculator.
4. The scantron and Part III will be collected at the end of the exam. You may take Part I and Part II with you at the end of the exam.

Here are some definitions/identities/integrals that might be useful:

$$\cosh t = \frac{e^t + e^{-t}}{2}$$
$$\sinh t = \frac{e^t - e^{-t}}{2}$$
$$e^{it} = \cos t + i \sin t$$
$$\cos t = \frac{e^{it} + e^{-it}}{2}$$
$$\sin t = \frac{e^{it} - e^{-it}}{2i}$$
$$\int \sec t dt = \ln|\sec t + \tan t| + C$$
$$\int \csc t dt = \ln|\csc t - \cot t| + C$$

Part I. Multiple Choices $5 \times 10 = 50$ points

1. Consider the 2nd order linear equation with constant coefficients:

$$y'' + ay' + by = 0.$$

If r_1 and r_2 are the roots of its characterisitic equation, then what is $r_1^2 + r_2^2$?

A. 1
B. $\sqrt{a^2 - 4b}$
C. $a^2 - 4b$
D. $a^2 + 2b$
E. $a^2 - 2b$
F. none of the above

E

The characterisitic equation is

$$r^2 + ar + b = (r - r_1)(r - r_2) = r^2 - (r_1 + r_2)r + r_1 r_2 = 0.$$

Therefore,

$$r_1 + r_2 = -a, \qquad r_1 r_2 = b$$
$$r_1^2 + r_2^2 = (r_1 + r_2)^2 - 2r_1 r_2 = a^2 - 2b.$$

2. Find the largest interval on which the following IVP has a unique solution.

$$(2t+1)y'' + \sin t \cdot y' + e^{t-3}y = \tan t, \qquad y(0) = 0, \quad y'(0) = 1.$$

A. $[0, \infty)$
B. $(-\frac{1}{2}, \infty)$
C. $(-\frac{\pi}{2}, \infty)$
D. $(-\frac{1}{2}, \frac{\pi}{2})$
E. $(-\pi, \pi)$
F. none of the above

D

Rewriting the equation, we have

$$y'' + \frac{\sin t}{2t+1} \cdot y' + \frac{e^{t-3}}{2t+1}y = \frac{\tan t}{2t+1}.$$

Discontinuities occur as $2t+1 = 0$ and $\tan t = 0$. Thus, the largest interval of continuity, containing 0, is $(-\frac{1}{2}, \frac{\pi}{2})$.

3. Consider the differential equation

$$y'' - y = 0. \tag{1}$$

Which of the following is NOT a solution of (1)?

A. 0

B. e^t

C. e^{-t}

D. $\cosh t$

E. $\cosh(t+1)$

F. all of the above are solutions of (1)

F

The general solution is

$$y(t) = c_1 e^t + c_2 e^{-t}.$$

We note that

$$\cosh(t+1) = \frac{e^{t+1} - e^{-(t+1)}}{2} = \frac{e}{2}e^t - \frac{1}{2e}e^{-t}.$$

4. Consider the differential equation

$$y'' + y = 0. \tag{2}$$

Which of the following is NOT a solution of (2)?

A. 0

B. $\sin t$

C. $\cos t$

D. $\cos(t+1)$

E. $\tan t$

F. all of the above are solutions of (2)

E

The general solution is

$$y(t) = c_1 \cos t + c_2 \sin t.$$

We note that

$$\cos(t+1) = \cos 1 \cos t - \sin 1 \sin t.$$

5. Find the general solution of

$$2y'' - 7y' + 3y = 0.$$

A. $y(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{3t}$

B. $y(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-3t}$

C. $y(t) = c_1 e^{\frac{1}{2}t} + c_2 e^{3t}$

D. $y(t) = c_1 e^{\frac{1}{2}t} + c_2$

E. $y(t) = c_1 + c_2 e^{-3t}$

F. none of the above

C

The characterisitic equation is

$$2r^2 - 7r + 3 = 0.$$

The roots are

$$r_1 = \frac{1}{2}, \qquad r_2 = 3.$$

$$r_1 + r_2 = -a, \qquad r_1 r_2 = b$$
$$r_1^2 + r_2^2 = (r_1 + r_2)^2 - 2r_1 r_2 = a^2 - 2b.$$

6. Find the solution to the IVP below:

$$y'' + 2y' + y = 0, \qquad y(0) = 5, \quad y'(0) = -3.$$

A. $5e^{-t} - 2te^{-t}$
B. $5e^{-t} + 2te^{-t}$
C. $5e^{-t} + 3e^{t}$
D. $5e^{-t} - 3te^{-t}$
E. $5e^{-t} - 2e^{t}$
F. none of the above

B

The characterisitic equation is

$$r^2 + 2r + 1 = 0.$$

We have repeated roots

$$r_1 = r_2 = -1,$$

and the general solution is

$$y(t) = c_1 e^{-t} + c_2 te^{-t}.$$

Taking derivative, we have

$$y'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 te^{-t}.$$

The initial condition implies

$$c_1 = 5, \qquad -c_1 + c_2 = -3,$$

so it follows that

$$c_1 = 5, \qquad c_2 = 2.$$

7. Compute the Wronskian $W(y_1, y_2)(t)$ for $t > 0$ where

$$y_1(t) = t \ln t, \qquad y_2(t) = t^2.$$

A. $t^2(\ln t + 1)$
B. $t^2(\ln t - 1)$
C. $t^2 \ln t$
D. $t \ln t + t^2$
E. 0
F. none of the above

B

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} t \ln t & t^2 \\ \ln t + 1 & 2t \end{vmatrix} \\ &= 2t^2 \ln t - t^2(\ln t + 1) \\ &= t^2(\ln t - 1). \end{aligned}$$

8. Consider the differential equation

$$3y'' + y' - 2y = 2\cos t \tag{3}$$

Which of the following is the general solution of (3)?

A. $c_1 e^{\frac{2}{3}t} + c_2 e^{-t} - \frac{5}{13}\cos t + \frac{1}{13}\sin t$
B. $c_1 e^{\frac{2}{3}t} + c_2 e^{-t} - \frac{5}{13}\cosh t + \frac{1}{13}\sinh t$
C. $c_1 e^{\frac{2}{3}t} + c_2 e^{t} + \frac{5}{26}e^{it} + \frac{1}{26}e^{-it}$
D. $c_1 e^{\frac{2}{3}t} + c_2 e^{t} - \frac{5}{13}\cos t + \frac{1}{13}\sin t$
E. $c_1 e^{\frac{2}{3}t} + c_2 e^{t} - \frac{5}{13}e^{t} + \frac{1}{13}e^{-t}$
F. none of the above

A

The general solution of the corresponding homogeneous equation is

$$y_c(t) = c_1 e^{\frac{2}{3}t} + c_2 e^{-t}.$$

To find a particular solution, we try

$$Y(t) = A\cos t + B\sin t.$$

Substituting into the original equation, we obtain

$$3(-A\cos t - B\sin t) + (-A\sin t + B\cos t) - 2(A\cos t + B\sin t) = 2\cos t,$$
$$(-5A + B)\cos t + (-A - 5B)\sin t = 2\cos t.$$

Thus, we have

$$-5A + B = 2 \quad -A - 5B = 0$$

and

$$A = -\frac{5}{13}, \qquad B = \frac{1}{13}.$$

9. Consider the differential equation

$$t^2 y'' - t(t+2)y' + (t+2)y = 0, \qquad t > 0. \tag{4}$$

The function $y_1(t) = t$ is a solution of (4). Choose the function y_2 such that y_1 and y_2 form a fundamental set of solutions to (4).

A. $y_2(t) = t^2$
B. $y_2(t) = e^t$
C. $y_2(t) = te^t$
D. $y_2(t) = t^2 e^t$
E. $y_2(t) = \ln t$
F. none of the above

C

To obtain the general solution, we try

$$y(t) = tv(t).$$

Substituting into the original equation, we obtain

$$v'' - v' = 0.$$

Writing $w = v'$, we obtain

$$w' - w = 0,$$

which means $v' = w = c_2 e^t$, and so it follows that

$$v(t) = c_2 e^t + c_1,$$

and

$$y(t) = tv(t) = c_2 te^t + c_1 t.$$

10. Consider the differential equation

$$y''' + y'' = 3e^t + 4t^2 \tag{5}$$

Which of the following is NOT a solution of (5)?

A. $\frac{3}{2}e^t + 4t^2 - \frac{4}{3}t^3 + \frac{1}{3}t^4$
B. $\frac{3}{2}e^t + 1 + 4t^2 - \frac{4}{3}t^3 + \frac{1}{3}t^4$
C. $\frac{3}{2}e^t + 1 + t + 4t^2 - \frac{4}{3}t^3 + \frac{1}{3}t^4$
D. $\frac{3}{2}e^t + 2015t + 4t^2 - \frac{4}{3}t^3 + \frac{1}{3}t^4$
E. $2\cosh t + 1 + t + 4t^2 - \frac{4}{3}t^3 + \frac{1}{3}t^4$
F. all of the above are solutions of (5)

E

From the format of the problem, we know that at most one of A, B, C, D and E, is not a solution of the equation.

The characterisitic equation of the corresponding homogeneous equation is

$$r^3 + r^2 = r^2(r+1) = 0,$$

so the general solution of the corresponding homogeneous equation is

$$y_c(t) = c_1 e^{-t} + c_2 + c_3 t.$$

Examing the solution, we see that A, B, C and D differ by a solution of the corresponding homogeneous equation. So if only one of A, B, C, D and E is not a a solution of (5), then it must be E.

The following is not strictly necessary.

To verify that

$$y(t) = \frac{3}{2}e^t + 4t^2 - \frac{4}{3}t^3 + \frac{1}{3}t^4$$

is a solution of (5), we use the undetermined coefficients and try

$$Y(t) = ae^t + bt^2 + ct^3 + dt^4.$$

Substituting it back into (5) and solve for a, b, c and d, we obtain

$$a = \frac{3}{2}, \qquad b = 4, \qquad c = -\frac{4}{3}, \qquad d = \frac{1}{3}.$$

Part II. True/False $5 \times 2 = 10$ points

Choose 'A' if the statement is true; choose 'B' if the statement is false.

11. For a second order linear homogeneous ordinary differential equation with constant coefficients, if the characteristic equation has no real roots, then we cannot solve the equation.

12. Let p and q be continuous functions, and let y_1 and y_2 be solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

If $W(y_1, y_2)(0) \neq 0$, then $W(y_1, y_2)(t) \neq 0$ for all t.

13. The functions $y_1, y_2, \ldots, y_n$ are linearly independent, if and only if

$$W(y_1, y_2, \ldots, y_n)(t_0) \neq 0$$

for some t_0.

14. The quasi-frenquency of the dampened spring is always smaller than the natural frequency.

15. There are continuous functions p and q such that

$$y_1(t) = t \ln t, \qquad y_2(t) = t^2$$

form a fundamental set of solutions of the second order homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0, \qquad t > 0.$$

Hint: You may use your result in 7.

B A B A B

13. This is only true if $y_1, y_2, \ldots, y_n$ are solutions of the homogeneous equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_{n-1}(t)y' + p_n(t)y = 0.$$

See Theorem 4.1.2 in Boyce-DiPrima.

15. If y_1 and y_2 form a fundamental set of solutions for the given equation, then by Abel's theorem, the Wronskian $W(y_1, y_2)$ is either always zero or never zero for $t > 0$. This is not the case, since from 7, we know

$$W(y_1, y_2)(t) = t^2(\ln t - 1).$$

Math 217 Exam 1 Sept 17, 2015

Part III will be collected separately. Please write down your name and your student number.
Student No.

Name:

For graders:

16.

17.

18

19.

Total:

Part III. Hand-graded problems $10 + 10 + 20 = 40$ points

16. (5+5 = 10 points)
Let m, γ and k be positive constant. Find the general solution of equation

$$mu''(t) + \gamma u'(t) + ku(t) = 0$$

in each of the two cases below:
(a) $\gamma^2 - 4km < 0$
(b) $\gamma^2 - 4km > 0$

The characteristic equation is

$$mr^2 + \gamma r + k = 0.$$

The roots are

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

If $\gamma^2 - 4km < 0$, then the roots are

$$r = -\frac{\gamma}{2m} \pm i\frac{\sqrt{4km - \gamma^2}}{2m} = -\frac{\gamma}{2m} \pm i\mu$$

where $\mu = \frac{\sqrt{4km-\gamma^2}}{2m}$. The general solution is

$$y(t) = e^{-\frac{\gamma t}{2m}} \left(A\cos(\mu t) + B\sin(\mu t)\right).$$

If $\gamma^2 - 4km > 0$, then the roots are

$$r_1 = \frac{-\gamma + \sqrt{\gamma^2 - 4km}}{2m}, \qquad r_1 = \frac{-\gamma - \sqrt{\gamma^2 - 4km}}{2m}.$$

The general solution is

$$y(t) = Ae^{r_1 t} + Be^{r_2 t}.$$

17. (10 points)
Find the general solution of the equation

$$t^2y'' + 2ty' - 12y = 0, \qquad t > 0 \tag{6}$$

We make the substituion $x = \ln t$.

———— The following is not strictly necessary ————

Then we have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx},$$

$$\frac{d^2y}{dt^2} = \frac{d\frac{dy}{dt}}{dx}\frac{dx}{dt} = \left(\frac{1}{t}\frac{d^2y}{dx^2} - \frac{1}{t^2}\frac{dt}{dx}\frac{dy}{dx}\right)\frac{1}{t} = \frac{1}{t^2}\frac{d^2y}{dx^2} - \frac{1}{t^2}\frac{dy}{dx}.$$

———— The above is not strictly necessary ————

It follows that

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0. \tag{7}$$

The characteristic equation $r^2 + r - 12 = 0$ has roots

$$r_1 = 3, \qquad r_2 = -4.$$

Thus, the general solution of (7) is

$$y(x) = c_1e^{3x} + c_2e^{-4x}.$$

Hence, the general solution of (6) is

$$y(t) = c_1t^3 + c_2t^{-4}.$$

18. (10 points)
Find the general solution of the equation

$$y^{(3)} + y' - 10y = 0.$$

The characeteristic equation is

$$r^3 + r - 10 = 0.$$

The possible rational roots are ± 1, ± 2, ± 5, ± 10. Indeed, 2 is a root. Thus, we have

$$r^3 + r - 10 = (r-2)(r^2 + 2r + 5) = 0.$$

The roots are

$$r_1 = 2, \qquad r_2 = -1 + 2i, \qquad r_2 = -1 - 2i.$$

Hence, the general solution is

$$y(t) = c_1 e^{2t} + e^{-t}(c_2 \cos 2t + c_3 \sin 2t).$$

19. (10 points)
Find the general solution of the equation

$$y'' + y = \tan t.$$

For the corresponding homogeneous equation $y'' + y = 0$,

$$y_1(t) = \cos t, \qquad y_2(t) = \sin t.$$

form a fundamental set of solutions. Indeed, we have

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1 \neq 0.$$

We use the variation of parameters:

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

Computing u_1 and u_2, we obtain

$$\begin{aligned} u_1(t) &= -\int \sin t \tan t dt \\ &= \int \cos t - \sec t dt \\ &= \sin t - \ln|\sec t + \tan t| + c_1, \end{aligned}$$

and

$$\begin{aligned} u_2(t) &= \int \cos t \tan t dt \\ &= \int \sin t dt \\ &= -\cos t + c_2. \end{aligned}$$

Hence, the general solution is

$$\begin{aligned} y(t) &= (\sin t - \ln|\sec t + \tan t| + c_1)\cos t + (-\cos t + c_2)\sin t \\ &= c_1 \cos t + c_2 \sin t - (\cos t)\ln|\sec t + \tan t|. \end{aligned}$$