Math 217 Exam 3 Nov 17, 2015

Instructions:

1. There are three parts in this exam. Part I is multiple choice, Part II is True/False, and Part III consists of hand-graded problems.
2. The total number of points is 100.
3. You may use a calculator.
4. The scantron and Part III will be collected at the end of the exam. You may take Part I and Part II with you at the end of the exam.

Here are some Taylor series that might be useful:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \dots$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

Part I. Multiple Choices $\qquad 5 \times 10 = 50$ points

1. Consider power series

$$\sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n.$$

From the power series below, choose the one that is different from the above.

A.

$$\sum_{n=1}^{\infty} \frac{n(n-1)}{2} x^n$$

B.

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^n$$

C.

$$\frac{1}{2}\sum_{n=0}^{\infty}(n-1)^2 x^n + \frac{1}{2}\sum_{n=1}^{\infty}(n-1)x^n$$

D.

$$\frac{1}{2}\sum_{n=0}^{\infty} n^2 x^{n+1} + \frac{1}{2}\sum_{n=1}^{\infty}(n-1)x^n$$

E.

$$\frac{1}{2}\sum_{n=0}^{\infty}(n+2)^2 x^{n+2} - \frac{1}{2}\sum_{n=1}^{\infty}(n+1)x^{n+1}$$

F. none of the above

2. For initial value problem:

$$(x^2-1)y'' + (x+1)y' - 2e^{x^2-1}y = 0, \qquad y(0) = 0, \quad y'(0) = -2,$$

if $y = \sum_{n=0}^{\infty} a_n x^n$ is the power series solution about 0, then we have...

A. $a_0 = 0$ and $a_1 = -2$

B. $a_0 = -2$ and $a_1 = 0$

C. $a_0 = 0$ and $a_1 = -\frac{1}{2}$

D. $a_0 = -\frac{1}{2}$ and $a_1 = 0$

E. $a_0 = -2$ and $a_1 = -2$

F. none of the above

3. For differential equation:

$$(x^2 - 1)y'' + (x + 1)y' - 2e^{x^2-1}y = 0, \tag{1}$$

if $y = \sum_{n=0}^{\infty} a_n x^n$ is the general power series solution about 0, then without calculating it explicitly, what is the lower bound of the radius of convergence?

A. 0

B. 1

C. $\frac{\pi}{2}$

D. 2π

E. ∞

F. none of the above

4. If we write $\cos[\ln(1+x)]$ as a power series about 0, that is,

$$\cos[\ln(1+x)] = \sum_{n=0}^{\infty} a_n x^n,$$

then what is the value of a_4?

A. 0
B. $-\frac{1}{3}$
C. $-\frac{1}{8}$
D. $\frac{1}{24}$
E. $-\frac{5}{12}$
F. none of the above

5. Which one of the following functions is NOT a solution of the differential equation

$$4x^2y'' - 8xy' + 9y = 0, \qquad x \neq 0. \tag{2}$$

A. $y(x) = e^{\frac{3}{2}\ln|x|}$

B. $y(x) = |x|^{\frac{3}{2}} \ln|x|$

C. $y(x) = e^{1+\frac{3}{2}\ln|x|+\ln(\ln|x|)}$

D. $y(x) = \ln\left(\frac{3}{2}|x|\right) \cdot |x|^{\frac{3}{2}}$

E. $y(x) = \left[e - \ln\left(|x|^{\frac{3}{2}}\right)\right] \cdot |x|^{\frac{3}{2}}$

F. all of the above are solutions of (2)

6. For the intial value problem:

$$2x^2y'' - 5xy' + 5y = 0, \qquad y(1) = 0, \quad y'(1) = \frac{3}{2},$$

find x_0 where $y'(x_0) = 0$.

A. 1
B. $\left(\frac{4}{25}\right)^{\frac{1}{3}}$
C. $\frac{5}{2}$
D. $\frac{3}{2}$
E. $\ln 5 - \ln 2$
F. none of the above

7. Find the solution for the intial value problem:

$$x^2y'' - xy' + 2y = 0, \qquad y(1) = 1, \quad y'(1) = 1.$$

A. $x + 1$

B. $x\cos(\ln x)$

C. $x\cos(\ln x) + x\sin(\ln x)$

D. $x\cos(\ln x) - x\sin(\ln x)$

E. x^{1+i}

F. none of the above

8. For the differential equation

$$(1-\cos x)\cdot y'' + (e^x - 1)\cdot y' + \frac{3}{2}y = 0, \tag{3}$$

0 is a regular singular point. Find the indicial equation about 0.

A. $r^2 - r + \frac{3}{2} = 0$
B. $r^2 + 2r - 3 = 0$
C. $r^2 + r - 3 = 0$
D. $r^2 - 4r + 4 = 0$
E. $r^2 - 3r + 2 = 0$
F. none of the above

9. Consider the power series

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n, \\ y_2(x) &= \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n. \end{aligned} \tag{4}$$

What is the Wronskian at 0? That is, what is $W(y_1, y_2)(0)$?

A. $a_0 b_1 - a_1 b_0$

B. $a_1 b_0 - a_0 b_1$

C. $\frac{a_0 b_0}{2} - \frac{a_1 b_1}{2}$

D. $\frac{a_0 b_0}{2} - \frac{a_1 b_1}{2}$

E. $\sum_{n=0}^{\infty} a_n b_{n+1} - a_{n+1} b_n$

F. none of the above

10. Consider the differential equation

$$x^2y'' + (6x + x^2)y' + xy = 0, \tag{5}$$

for which 0 is regular singular point. To solve (5) by power series, we should begin by finding the coefficients of a Frobenius series. Of the Frobenius series below, which one is the correct trial solution?

A.

$$\sum_{n=1}^{\infty} a_n x^n$$

B.

$$1 + \sum_{n=1}^{\infty} a_n x^n$$

C.

$$\sum_{n=1}^{\infty} a_n x^{n-5}$$

D.

$$|x|^{-5}\left(1 + \sum_{n=1}^{\infty} a_n x^n\right)$$

E.

$$|x|^{-5}\ln|x|\left(1 + \sum_{n=1}^{\infty} a_n x^n\right)$$

F. none of the above

Part II. True/False $5 \times 2 = 10$ points

Choose 'A' if the statement is true; choose 'B' if the statement is false.

11. For a second order linear homogeneous ordinary differential equation with constant coefficients, there is no singular points.

12. For two convergent power series

$$y_1 = \sum_{n=0}^{\infty} a_n x^n, \qquad y_2 = \sum_{n=0}^{\infty} b_n x^n,$$

the Wronskian $W(y_1, y_2)$ is never zero.

13. For a second order linear homogeneous ordinary differential equation, we can always find two linearly independent power series solutions about an ordinary point.

14. If x_0 is a regular singular point of the differential equation

$$y'' + p(x)y' + q(x)y = 0,$$

and the indicial equation has real roots, then the equation has at least one Frobenius series solution about x_0.

15. For the power series

$$\sum_{n=0}^{\infty} a_n x^n,$$

if the radius of convergence is ρ and $|x_1| > \rho$, then the series

$$\sum_{n=0}^{\infty} a_n (x_1)^n$$

does not converge.

Math 217 Exam 3 Nov 17, 2015

Part III will be collected separately. Please write your NAME and your STUDENT NUMBER.
Student Number:

Name:

For graders:

16.

17.

18

19.

Total:

Here are some Taylor series that might be useful:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \dots$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

Part III. Hand-graded problems $10 + 10 + 20 = 40$ points

16. (10 points)
For the initial value problem:

$$y'' = y' + y, \qquad y(0) = 0, \qquad y'(0) = 1,$$

the point 0 is an ordianary point. Show that the power series solution about 0 is given by

$$y = \sum_{n=0}^{\infty} \frac{f_n}{n!} x^n,$$

where $\{f_n\}_{n=0}^{\infty}$ is the Fibonacci numbers defined by $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$.

This page is intentionally left blank.

17. (10 points)

The Hermite function

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2} \right)$$

is a polynomial of degree n. Compute H_4 explicitly.

Hint: It is difficult to expend e^x as a power series and to calculate it term by term. Try something else.

This page is intentionally left blank.

18. (5 + 5 + 5 + 5 = 20 points)
For the differential equation

$$2x^2y'' + 3xy' - (x^2 + 1)y = 0, \tag{6}$$

the point 0 is a regular singular point.

(a) Show that the roots of the indicial equation are $r_1 = \frac{1}{2}$ and $r_2 = -1$.

(b) Consider the Frobenius series solution

$$y(x) = |x|^r \sum_{n=0}^{\infty} a_n x^n.$$

Show that the recurrence relation is

$$a_n = \frac{a_{n-2}}{2(n+r)^2 + (n+r) - 1}, \qquad n \geq 2,$$

and $a_{2n+1} = 0$.

This page is intentionally left blank.

(c) For $r_1 = \frac{1}{2}$, compute the Frobenius solution y_1 up to $|x|^{\frac{1}{2}}x^6$.

(d) For $r_2 = -1$, compute the Frobenius solution y_2 up to $|x|^{-1}x^6$.

This page is intentionally left blank.

This page is intentionally left blank.