Math 217 Exam 3 Nov 17, 2015

Instructions:

1. There are three parts in this exam. Part I is multiple choice, Part II is True/False, and Part III consists of hand-graded problems.
2. The total number of points is 100.
3. You may use a calculator.
4. The scantron and Part III will be collected at the end of the exam. You may take Part I and Part II with you at the end of the exam.

Here are some Taylor series that might be useful:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \dots$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

Part I. Multiple Choices $5 \times 10 = 50$ points

1. Consider power series

$$\sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n.$$

From the power series below, choose the one that is different from the above.

A.

$$\sum_{n=1}^{\infty} \frac{n(n-1)}{2} x^n$$

B.

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^n$$

C.

$$\frac{1}{2}\sum_{n=0}^{\infty} (n-1)^2 x^n + \frac{1}{2}\sum_{n=1}^{\infty} (n-1) x^n$$

D.

$$\frac{1}{2}\sum_{n=0}^{\infty} n^2 x^{n+1} + \frac{1}{2}\sum_{n=1}^{\infty} (n-1) x^n$$

E.

$$\frac{1}{2}\sum_{n=0}^{\infty} (n+2)^2 x^{n+2} - \frac{1}{2}\sum_{n=1}^{\infty} (n+1) x^{n+1}$$

F. none of the above

C

The power series in C has an extra $\frac{1}{2}$ from the first sum.

Clearly, A and B are the same as the original sum. For C, D and E, we rewrite the original sum as follows:

$$\sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n = \sum_{n=0}^{\infty} \frac{(n-1)^2}{2} x^n + \sum_{n=0}^{\infty} \frac{n-1}{2} x^n$$
$$= \sum_{n=0}^{\infty} \frac{n^2}{2} x^n - \sum_{n=0}^{\infty} \frac{n}{2} x^n.$$

Now we may shift the indices to obtain the forms in C, D and E. It turns out that the first sum in C should begin with $n = 1$, which means that the answer in C has an extra $\frac{1}{2}$.

2. For initial value problem:

$$(x^2-1)y'' + (x+1)y' - 2e^{x^2-1}y = 0, \qquad y(0) = 0, \quad y'(0) = -2,$$

if $y = \sum_{n=0}^{\infty} a_n x^n$ is the power series solution about 0, then we have...

A. $a_0 = 0$ and $a_1 = -2$
B. $a_0 = -2$ and $a_1 = 0$
C. $a_0 = 0$ and $a_1 = -\frac{1}{2}$
D. $a_0 = -\frac{1}{2}$ and $a_1 = 0$
E. $a_0 = -2$ and $a_1 = -2$
F. none of the above

A

If $y = \sum_{n=0}^{\infty} a_n x^n$, then

$$y(0) = a_0 = 0$$

and

$$y'(0) = a_1 = -2.$$

3. For differential equation:

$$(x^2-1)y'' + (x+1)y' - 2e^{x^2-1}y = 0, \tag{1}$$

if $y = \sum_{n=0}^{\infty} a_n x^n$ is the general power series solution about 0, then without calculating it explicitly, what is the lower bound of the radius of convergence?

A. 0

B. 1

C. $\frac{\pi}{2}$

D. 2π

E. ∞

F. none of the above

B

Rewriting (1), we obtain

$$y'' + \frac{1}{x-1}y' - \frac{2e^{x^2-1}}{x^2-1}y = 0,$$

Since $\frac{1}{x-1}$ is analytic for $x < 1$ and $\frac{2e^{x^2-1}}{x^2-1}$ is analytic for $-1 < x < 1$, it follows that the general power series solution about 0 has radius of convergence at least 1.

4. If we write $\cos[\ln(1+x)]$ as a power series about 0, that is,

$$\cos[\ln(1+x)] = \sum_{n=0}^{\infty} a_n x^n,$$

then what is the value of a_4?

A. 0
B. $-\frac{1}{3}$
C. $-\frac{1}{8}$
D. $\frac{1}{24}$
E. $-\frac{5}{12}$
F. none of the above

E

$$\begin{aligned}
\cos[\ln(1+x)] &= 1 - \frac{\ln^2(1+x)}{2!} + \frac{\ln^4(1+x)}{4!} - \dots \\
&= 1 - \frac{1}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right)^2 + \frac{1}{24}(x - \dots)^4 - \dots \\
&= 1 + \dots - \frac{1}{2}\left(-\frac{x^2}{2}\right)^2 - \frac{1}{2}\cdot 2x \cdot \frac{x^3}{3} + \frac{1}{24}x^4 + \dots \\
&= 1 + \dots + \left(-\frac{1}{8} - \frac{1}{3} + \frac{1}{24}\right)x^4 + \dots \\
&= 1 + \dots - \frac{5}{12}x^4 + \dots
\end{aligned}$$

5. Which one of the following functions is NOT a solution of the differential equation

$$4x^2y'' - 8xy' + 9y = 0, \qquad x \neq 0. \tag{2}$$

A. $y(x) = e^{\frac{3}{2}\ln|x|}$
B. $y(x) = |x|^{\frac{3}{2}}\ln|x|$
C. $y(x) = e^{1+\frac{3}{2}\ln|x|+\ln(\ln|x|)}$
D. $y(x) = \ln\left(\frac{3}{2}|x|\right)\cdot|x|^{\frac{3}{2}}$
E. $y(x) = \left[e - \ln\left(|x|^{\frac{3}{2}}\right)\right]\cdot|x|^{\frac{3}{2}}$
F. all of the above are solutions of (2)

F

The indicial equation of (2) is

$$\begin{aligned} 4r(r-1) - 8r + 9 &= 0, \\ 4r^2 - 12r + 9 &= 0. \end{aligned}$$

We have repeated roots

$$r_1 = r_2 = \frac{3}{2}.$$

Hence the general solution is

$$y(x) = (c_1 + c_2\ln|x|)|x|^{\frac{3}{2}}.$$

Now you can verify that all of A, B, C, D and E are solutions.

6. For the intial value problem:

$$2x^2y'' - 5xy' + 5y = 0, \qquad y(1) = 0, \quad y'(1) = \frac{3}{2},$$

find x_0 where $y'(x_0) = 0$.

A. 1
B. $\left(\frac{4}{25}\right)^{\frac{1}{3}}$
C. $\frac{5}{2}$
D. $\frac{3}{2}$
E. $\ln 5 - \ln 2$
F. none of the above

B

The indicial equation of (2) is

$$\begin{aligned} 2r(r-1) - 5r + 5 &= 0, \\ 2r^2 - 7r + 5 &= 0. \end{aligned}$$

We have the roots

$$r_1 = 1, \qquad r_2 = \frac{5}{2},$$

and the general solution

$$y(x) = c_1 x + c_2 x^{\frac{5}{2}}.$$

The initial condition implies

$$c_1 + c_2 = 0, \qquad c_1 + \frac{5}{2}c_2 = \frac{3}{2},$$

so we have $c_1 = -1$, $c_2 = 1$, and

$$y(x) = -x + x^{\frac{5}{2}}.$$

Taking derivative, we get

$$y'(x) = -1 + \frac{5}{2}x^{\frac{3}{2}},$$

so it follows that $x_0 = \left(\frac{4}{25}\right)^{\frac{1}{3}}$.

7. Find the solution for the intial value problem:

$$x^2 y'' - xy' + 2y = 0, \qquad y(1) = 1, \quad y'(1) = 1.$$

A. $x + 1$

B. $x \cos(\ln x)$

C. $x \cos(\ln x) + x \sin(\ln x)$

D. $x \cos(\ln x) - x \sin(\ln x)$

E. x^{1+i}

F. none of the above

B

The indicial equation of (2) is

$$\begin{aligned} r(r-1) - r + 2 &= 0, \\ r^2 - 2r + 2 &= 0. \end{aligned}$$

We have the roots

$$r_{1,2} = 1 \pm i,$$

and the general solution

$$y(x) = x \left[c_1 \cos(\ln x) + c_2 \sin(\ln x) \right].$$

Taking derivative, we get

$$\begin{aligned} y'(x) &= c_1 \cos(\ln x) + c_2 \sin(\ln x) - c_1 \sin(\ln x) + c_2 \cos(\ln x) \\ &= (c_1 + c_2) \cos(\ln x) + (-c_1 + c_2) \sin(\ln x). \end{aligned}$$

The initial condition implies

$$c_1 = 1, \qquad c_1 + c_2 = 1,$$

so we have $c_1 = 1$ and $c_2 = 0$.

8. For the differential equation

$$(1-\cos x)\cdot y'' + (e^x - 1)\cdot y' + \frac{3}{2}y = 0, \tag{3}$$

0 is a regular singular point. Find the indicial equation about 0.

A. $r^2 - r + \frac{3}{2} = 0$
B. $r^2 + 2r - 3 = 0$
C. $r^2 + r - 3 = 0$
D. $r^2 - 4r + 4 = 0$
E. $r^2 - 3r + 2 = 0$
F. none of the above

F

Rewriting (3), we get

$$y'' + \frac{e^x - 1}{1-\cos x}\cdot y' - \frac{3}{2-2\cos x}\cdot y = 0.$$

It follows that

$$\begin{aligned} p_0 &= \lim_{x\to 0} \frac{x(e^x-1)}{1-\cos x} \\ &= \lim_{x\to 0} \frac{e^x - 1 + xe^x}{\sin x} \\ &= \lim_{x\to 0} \frac{e^x + e^x + xe^x}{\cos x} \\ &= 2, \end{aligned}$$

and

$$\begin{aligned} q_0 &= \lim_{x\to 0} \frac{3x^2}{2-2\cos x} \\ &= \lim_{x\to 0} \frac{6x}{2\sin x} \\ &= \lim_{x\to 0} \frac{6}{2\cos x} \\ &= 3. \end{aligned}$$

Thus, the indicial equation is

$$\begin{aligned} r(r-1) + 2r + 3 &= 0, \\ r^2 + r + 3 &= 0. \end{aligned}$$

9. Consider the power series

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n, \\ y_2(x) &= \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n. \end{aligned} \tag{4}$$

What is the Wronskian at 0? That is, what is $W(y_1, y_2)(0)$?

A. $a_0 b_1 - a_1 b_0$
B. $a_1 b_0 - a_0 b_1$
C. $\frac{a_0 b_0}{2} - \frac{a_1 b_1}{2}$
D. $\frac{a_0 b_0}{2} - \frac{a_1 b_1}{2}$
E. $\sum_{n=0}^{\infty} a_n b_{n+1} - a_{n+1} b_n$
F. none of the above

A

In fact, we just need the first two terms of (4):

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + \dots, \\ y_2(x) &= b_0 + b_1 x + \dots, \end{aligned}$$

which implies that

$$\begin{aligned} y_1(0) &= a_0, \qquad y_1'(0) = a_1, \\ y_2(0) &= b_0, \qquad y_2'(0) = b_1. \end{aligned}$$

Thus,

$$W(y_1, y_2)(0) = \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} = a_0 b_1 - a_1 b_0.$$

10. Consider the differential equation

$$x^2y'' + (6x + x^2)y' + xy = 0, \tag{5}$$

for which 0 is regular singular point. To solve (5) by power series, we should begin by finding the coefficients of a Frobenius series. Of the Frobenius series below, which one is the correct trial solution?

A.

$$\sum_{n=1}^{\infty} a_n x^n$$

B.

$$1 + \sum_{n=1}^{\infty} a_n x^n$$

C.

$$\sum_{n=1}^{\infty} a_n x^{n-5}$$

D.

$$|x|^{-5}\left(1 + \sum_{n=1}^{\infty} a_n x^n\right)$$

E.

$$|x|^{-5}\ln|x|\left(1 + \sum_{n=1}^{\infty} a_n x^n\right)$$

F. none of the above

B

It is easy to see that $p_0 = 6$ and $q_0 = 0$, so the indicial equation is

$$r(r-1) + 6r = r^2 + 5r = 0.$$

The two roots are $r_1 = 0$ and $r_2 = -5$, where 0 is the bigger root. Now, we follow the recipe outlined in Theorem 5.6.1 of Boyce-DiPrima.

Part II. True/False $5 \times 2 = 10$ points

Choose 'A' if the statement is true; choose 'B' if the statement is false.

11. For a second order linear homogeneous ordinary differential equation with constant coefficients, there is no singular points.

12. For two convergent power series

$$y_1 = \sum_{n=0}^{\infty} a_n x^n, \qquad y_2 = \sum_{n=0}^{\infty} b_n x^n,$$

the Wronskian $W(y_1, y_2)$ is never zero.

13. For a second order linear homogeneous ordinary differential equation, we can always find two linearly independent power series solutions about an ordinary point.

14. If x_0 is a regular singular point of the differential equation

$$y'' + p(x)y' + q(x)y = 0,$$

and the indicial equation has real roots, then the equation has at least one Frobenius series solution about x_0.

15. For the power series

$$\sum_{n=0}^{\infty} a_n x^n,$$

if the radius of convergence is ρ and $|x_1| > \rho$, then the series

$$\sum_{n=0}^{\infty} a_n (x_1)^n$$

does not converge.

A B A A A

Math 217 Exam 3   Nov 17, 2015

Part III will be collected separately. Please write your NAME and your STUDENT NUMBER.
Student Number:

Name:

For graders:

16.

17.

18

19.

Total:

Here are some Taylor series that might be useful:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \dots$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

Part III. Hand-graded problems $10 + 10 + 20 = 40$ points

16. (10 points)
For the initial value problem:

$$y'' = y' + y, \qquad y(0) = 0, \qquad y'(0) = 1,$$

the point 0 is an ordianary point. Show that the power series solution about 0 is given by

$$y = \sum_{n=0}^{\infty} \frac{f_n}{n!} x^n,$$

where $\{f_n\}_{n=0}^{\infty}$ is the Fibonacci numbers defined by $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$.

Writing

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

we have

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}.$$

Substituting these back into the differential equation, we get

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n,$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) x^n = \sum_{n=0}^{\infty} a_{n+1}(n+1) x^n + \sum_{n=0}^{\infty} a_n x^n,$$

which implies

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)} + \frac{a_{n+1}}{n+2}.$$

The initial condition implies

$$a_0 = 0 = \frac{f_0}{0!},$$
$$a_1 = 1 = \frac{f_1}{1!}.$$

Assuming $a_m = \frac{f_m}{m!}$ for $m \leq n+1$, we get that

$$\begin{aligned} a_{n+2} &= \frac{a_n}{(n+2)(n+1)} + \frac{a_{n+1}}{n+2} \\ &= \frac{f_n}{n!} \frac{1}{(n+2)(n+1)} + \frac{f_{n+1}}{(n+1)!} \frac{1}{n+2} \\ &= \frac{f_n + f_{n+1}}{(n+2)!} \\ &= \frac{f_{n+2}}{(n+2)!}. \end{aligned}$$

This page is intentionally left blank.

17. (10 points)
The Hermite function

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2} \right)$$

is a polynomial of degree n. Compute H_4 explicitly.

Hint: It is difficult to expend e^x as a power series and to calculate it term by term. Try something else.

$$\begin{aligned}
H_4(x) &= e^{x^2} \frac{d^4}{dx^4} \left(e^{-x^2} \right) \\
&= e^{x^2} \frac{d^3}{dx^3} \left(-2xe^{-x^2} \right) \\
&= e^{x^2} \frac{d^2}{dx^2} \left((4x^2 - 2)e^{-x^2} \right) \\
&= e^{x^2} \frac{d}{dx} \left[\left((4x^2 - 2)(-2x) + 8x \right) e^{-x^2} \right] \\
&= e^{x^2} \frac{d}{dx} \left((-8x^3 + 12x)e^{-x^2} \right) \\
&= e^{x^2} \left[(-8x^3 + 12x)(-2x) - 24x^2 + 12 \right] e^{-x^2} \\
&= 16x^4 - 48x^2 + 12.
\end{aligned}$$

This page is intentionally left blank.

18. (5 + 5 + 5 + 5 = 20 points)
For the differential equation

$$2x^2y'' + 3xy' - (x^2 + 1)y = 0, \tag{6}$$

the point 0 is a regular singular point.

(a) Show that the roots of the indicial equation are $r_1 = \frac{1}{2}$ and $r_2 = -1$.

Rewriting (6), we get

$$y'' + \frac{3}{2x} \cdot y' - \frac{x^2 + 1}{2x^2} \cdot y = 0.$$

It follows that

$$xp(x) = x \cdot \frac{3}{2x} = \frac{3}{2},$$
$$xq(x) = x^2 \cdot \left(-\frac{x^2 + 1}{2x^2}\right) = -\frac{1}{2} - \frac{x^2}{2}.$$

Thus, the indicial equation is

$$r(r-1) + \frac{3}{2}r - \frac{1}{2} = 0,$$
$$2r^2 + r - 1 = 0$$

where the roots are $r_1 = \frac{1}{2}$ and $r_2 = -1$.

(b) Consider the Frobenius series solution

$$y(x) = |x|^r \sum_{n=0}^{\infty} a_n x^n.$$

Show that the recurrence relation is

$$a_n = \frac{a_{n-2}}{2(n+r)^2 + (n+r) - 1}, \qquad n \geq 2,$$

and $a_{2n+1} = 0$.

For simplicity, we assume $x > 0$ to drop the absolute value. We begin with

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^{n+r}, \\
y'(x) &= \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \\
y''(x) &= \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}.
\end{aligned}$$

Substituting into (6), we get

$$\begin{aligned}
2\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r} + 3\sum_{n=0}^{\infty} a_n (n+r)x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} a_n x^{n+r} &= 0, \\
2\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r} + 3\sum_{n=0}^{\infty} a_n (n+r)x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} &= 0, \\
[2r(r-1) + 3r - 1]a_0 x^r + [2(r+1)r + 3(r+1) - 1]a_1 x^{r+1} & \\
+ \sum_{n=0}^{\infty} \{[2(n+r)(n+r-1) + 3(n+r) - 1]a_n - a_{n-2}\} x^{n+r} &= 0, \\
(2r^2 + r - 1)a_0 x^r + (2r^2 + 5r + 2)a_1 x^{r+1} + \sum_{n=0}^{\infty} \{[2(n+r)^2 + (n+r) - 1]a_n - a_{n-2}\} x^{n+r} &= 0.
\end{aligned}$$

Thus,

$$\begin{gathered}
(2r^2 + r - 1)a_0 = 0, \\
(2r^2 + 5r + 2)a_1 = 0, \\
[2(n+r)^2 + (n+r) - 1]a_n - a_{n-2} = 0, \qquad n \geq 2,
\end{gathered}$$

and the recurrence relation

$$a_n = \frac{a_{n-2}}{2(n+r)^2 + (n+r) - 1}, \qquad n \geq 2$$

follows. Since $a_0 \neq 0$, we have the indicial equation $2r^2 + r - 1 = 0$. For both $r_1 = \frac{1}{2}$ and $r_2 = -1$, we have

$$2r^2 + 5r + 2 \neq 0.$$

It follows that $a_1 = 0$, and therefore $a_{2n+1} = 0$.

This page is intentionally left blank.

(c) For $r_1 = \frac{1}{2}$, compute the Frobenius solution y_1 up to $|x|^{\frac{1}{2}}x^6$.

For $r_1 = \frac{1}{2}$, the recurrence relation is

$$a_n = \frac{a_{n-2}}{2\left(n+\frac{1}{2}\right)^2 + \left(n+\frac{1}{2}\right) - 1} = \frac{a_{n-2}}{n(2n+3)}, \qquad n \geq 2.$$

Thus,

$$a_2 = \frac{a_0}{2 \cdot 7} = \frac{a_0}{14},$$
$$a_4 = \frac{a_2}{4 \cdot 11} = \frac{a_0}{616},$$
$$a_6 = \frac{a_4}{6 \cdot 15} = \frac{a_0}{55440},$$

and it follows that

$$y_1(x) = a_0|x|^{\frac{1}{2}}\left(1 + \frac{x^2}{14} + \frac{x^4}{616} + \frac{x^6}{55440} + \dots\right).$$

(d) For $r_2 = -1$, compute the Frobenius solution y_2 up to $|x|^{-1}x^6$.

For $r_2 = -1$, to distinguish from the case $r_1 = \frac{1}{2}$, we write b_n in place of a_n, so the recurrence relation is

$$b_n = \frac{b_{n-2}}{2\,(n-1)^2 + (n-1) - 1} = \frac{b_{n-2}}{n(2n-3)}, \qquad n \geq 2.$$

Thus,

$$\begin{aligned} b_2 &= \frac{b_0}{2 \cdot 1} = \frac{b_0}{2}, \\ b_4 &= \frac{b_2}{4 \cdot 5} = \frac{b_0}{40}, \\ b_6 &= \frac{b_4}{6 \cdot 9} = \frac{b_0}{2160}, \end{aligned}$$

and it follows that

$$y_2(x) = b_0|x|^{-1}\left(1 + \frac{x^2}{2} + \frac{x^4}{40} + \frac{x^6}{2160} + \dots\right).$$