Math 217 Final Exam Dec 11, 2015

Instructions:

1. There are three parts in this exam. Part I is multiple choice, Part II is True/False, and Part III consists of hand-graded problems.
2. The total number of points is 100.
3. You may use a calculator.
4. The scantron and Part III will be collected at the end of the exam. You may take Part I and Part II with you at the end of the exam.

Taylor series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \dots$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

Trigonometry:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin(x+y) = \sin x \cos y + \sin y \cos x$$

$$2\cos x \cos y = \cos(x-y) + \cos(x+y)$$

$$2\sin x \sin y = \cos(x-y) - \cos(x+y)$$

$$2\sin x \cos y = \sin(x+y) - \sin(x-y)$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\cosh t = \frac{e^t + e^{-t}}{2}$$

$$\sinh t = \frac{e^t - e^{-t}}{2}$$

$$e^{it} = \cos t + i \sin t$$

Laplace transforms:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
$(-t)^n f(t)$	$F^{(n)}(s)$

Part I. Multiple Choices $5 \times 10 = 50$ points

1. Find the solution of the initial value problem:

$$(1+x)y' + y = \cos x, \qquad y(0) = 1. \tag{1}$$

A. $y = 1 + x$
B. $y = \frac{\cos x}{1+x}$
C. $y = \frac{\sin x}{1+x}$
D. $y = \frac{1+\sin x}{1+x}$
E. $y = \frac{e^x}{1+x}$
F. none of the above

2. Find the (implicit) solution of the exact differential equation

$$(x+y)y' = -(x+\arctan y)(1+y^2), \qquad y(0)=0. \tag{2}$$

A. $\frac{x^2}{2} + x\arctan y = 0$

B. $\frac{x^2}{2} + x\arctan y + \frac{1}{2}\ln(1+y^2) = 0$

C. $y = \tan\left(-\frac{x}{2}\right)$

D. $y^2 = e^{-x^2} - 1$

E. $y = e^{-x^2} - 1$

F. none of the above

3. Find the solution to the initial value problem:

$$2y''' - 3y'' - 2y' = 0, \qquad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 3. \tag{3}$$

A. $-\frac{7}{2} + 4e^{-\frac{1}{2}t} + \frac{1}{2}e^{2t}$

B. $1 - 2e^{-\frac{1}{2}t} + 2e^{2t}$

C. $\cosh t + \frac{1}{2}\sinh(2t)$

D. $\cos t + \frac{1}{2}\sin(2t)$

E. $e^{it} + 3e^{-it}$

F. none of the above

4. Which of the following is NOT a solution of the following differential equation?

$$y'' + 4y = \cos 3t \tag{4}$$

A. $-\frac{1}{5}\cos 3t$

B. $\cos 2t - \frac{1}{5}\cos 3t$

C. $\sin 2t - \frac{1}{5}\cos 3t$

D. $e^{2ti} + e^{-2ti} - \frac{1}{5}\cos 3t$

E. $\cos 2t + \sin 2t$

F. all of the above are solutions of (4)

5. For the differential equation (5), what is the lower bound of the radius of convergence for the power series solution around 0?

$$(1+x^2)y'' - \cos x \cdot y' + e^x y = 0. \tag{5}$$

A. 0

B. 1

C. $\sqrt{2}$

D. π

E. ∞

F. none of the above

6. If

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

is the Taylor series around 0 for the function

$$\cos(1+x) + \sin(1+x),$$

then what is the value of a_{2015}?

A.

$$a_{2015} = \frac{\sin 1 - \cos 1}{2015}$$

B.

$$a_{2015} = \frac{\cos 1 - \sin 1}{2015!}$$

C.

$$a_{2015} = \frac{\sin 1 - \cos 1}{2015!}$$

D.

$$a_{2015} = \frac{\pi}{2015!}$$

E.

$$a_{2015} = \frac{\pi^{2015}}{2015!}$$

F. none of the above

7. Find the inverse Laplace transform of

$$F(s) = \frac{e^{2-2s}(2s^2 + \pi^2 - 1)}{(s-1)(s^2+\pi^2)}.$$

A.

$$f(t) = e^t + e^2 \cos(\pi t) + \frac{e^2}{\pi} \sin(\pi t)$$

B.

$$f(t) = u_2(t) \left[e^t + e^2 \cos(\pi t) + \frac{e^2}{\pi} \sin(\pi t) \right]$$

C.

$$f(t) = u_2(t) \left[e^{t-2} + e^2 \cos(\pi t) + \frac{e^2}{\pi} \sin(\pi t) \right]$$

D.

$$f(t) = u_2(t) \left[e^t + e^2 \cos(t-2) + \frac{e^2}{\pi} \sin(t-2) \right]$$

E.

$$f(t) = u_2(t) \left[e^{t-2} + e^2 \cos(t-2) + e^2 \sin(t-2) \right]$$

F. none of the above

8. Find the Laplace transform of the piecewise continuous function

$$f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 2 - t & \text{if } 1 \le t < 2 \\ 0 & \text{if } t \ge 2 \end{cases} \tag{6}$$

A.

$$F(s) = \frac{1}{s^2} - \frac{2}{s^2} + \frac{1}{s^2}$$

B.

$$F(s) = 1 - e^{-s} + e^{-2s}$$

C.

$$F(s) = 1 - 2e^{-s} + e^{-2s}$$

D.

$$F(s) = \frac{1}{s^2} + \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$$

E.

$$F(s) = \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$$

F. none of the above

9. If $y(t)$ is the solution of the initial value problem:

$$y'' + 4y = \delta(t - 2014\pi), \qquad y(0) = 0, \quad y'(0) = 0, \tag{7}$$

then what is the value of $y(2015\pi)$?

A. 0
B. -1
C. 1
D. π
E. $\frac{\pi}{2}$
F. none of the above

10. Find the inverse Laplace transform of

$$F(s) = \ln\left(\frac{s^2+1}{s^2+4}\right). \tag{8}$$

A.

$$f(t) = e^{t^2+1} + e^{t^2+4}$$

B.

$$f(t) = e^{2t} + e^{-2t}$$

C.

$$f(t) = u_1(t)\cos t + u_2(t)\sin(2t)$$

D.

$$f(t) = \frac{2}{t}(2\cos t + 1)(\cos t - 1)$$

E.

$$f(t) = \frac{2}{t}(\cos t - \cos(2t))$$

F. none of the above

Part II. True/False $5 \times 2 = 10$ points

Choose A if the statement is true; choose B if the statement is false.

11. There is a unique solution to the initial value problem:

$$y' - y^{\frac{1}{2}} = 0, \qquad y(0) = 0.$$

12. If p and q are continuous functions, and y_1 and y_2 are solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

then we have

$$W(y_1, y_2)(t) = e^{-\int_{t_0}^{t} p(\tau)d\tau}.$$

for some constant t_0.

13. If p and q are continuous real-valued functions, and $u(t)+iv(t)$ are complex-valued solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0, \tag{9}$$

then both u and v are also solutions of (9).

14. For the power series

$$\sum_{n=0}^{\infty} a_n x^n,$$

if the radius of convergence is ρ and $|x_1| \geq \rho$, then the series

$$\sum_{n=0}^{\infty} a_n (x_1)^n$$

does not converge.

15. If f is continuous function on $[0, \infty)$, then the Laplace transform

$$\mathcal{L}\{f(t)\} = F(s)$$

exists for $s > 0$.

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Part III will be collected separately. Please write your NAME and your STUDENT NUMBER.

Student Number:

Name:

For graders:

16a + 16b. + 16c. + 16d.

16e.

17.

Total:

Part III. Hand-graded problems $10 + 10 + 20 = 40$ points

16a. (4 points)
Find the general solution of the homogenous differential equation:

$$y'' - 4y = 0. \tag{10}$$

16b. (6 points) Use the method of undetermined coefficients to find the general solution of the non-homogeneous differential equation:

$$y'' - 4y = \sinh 2t. \tag{11}$$

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16c. (6 points)

Use the method of variation of parameters to find the general solution of the non-homogeneous differential equation:

$$y'' - 4y = \sinh 2t. \tag{12}$$

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16d. (4 points)
Use the results in 16b. or 16c. to find the unique solution of the initial value problem:

$$y'' - 4y = \sinh 2t, \qquad y(0) = 0, \quad y'(0) = 0. \tag{13}$$

16e. (10 points)
Use the method of Laplace transform to find the solution of the initial value problem:

$$y'' - 4y = \sinh 2t, \qquad y(0) = 0, \quad y'(0) = 0. \tag{14}$$

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17. Find the solution of the initial value problem:

$$y'' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 1 \tag{15}$$

where

$$f(t) = \begin{cases} 1 & \text{if } 0 \leq t < \pi \\ 0 & \text{if } t \geq \pi \end{cases} \tag{16}$$

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