Math 217 Final Exam Dec 11, 2015

Instructions:

1. There are three parts in this exam. Part I is multiple choice, Part II is True/False, and Part III consists of hand-graded problems.
2. The total number of points is 100.
3. You may use a calculator.
4. The scantron and Part III will be collected at the end of the exam. You may take Part I and Part II with you at the end of the exam.

Taylor series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \dots$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

Trigonometry:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin(x+y) = \sin x \cos y + \sin y \cos x$$

$$2 \cos x \cos y = \cos(x-y) + \cos(x+y)$$

$$2 \sin x \sin y = \cos(x-y) - \cos(x+y)$$

$$2 \sin x \cos y = \sin(x+y) - \sin(x-y)$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\cosh t = \frac{e^t + e^{-t}}{2}$$

$$\sinh t = \frac{e^t - e^{-t}}{2}$$

$$e^{it} = \cos t + i \sin t$$

Laplace transforms:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
$(-t)^n f(t)$	$F^{(n)}(s)$

Part I. Multiple Choices $5 \times 10 = 50$ points

1. Find the solution of the initial value problem:

$$(1+x)y' + y = \cos x, \qquad y(0) = 1. \tag{1}$$

A. $y = 1 + x$
B. $y = \frac{\cos x}{1+x}$
C. $y = \frac{\sin x}{1+x}$
D. $y = \frac{1+\sin x}{1+x}$
E. $y = \frac{e^x}{1+x}$
F. none of the above

D

From (1), we have

$$\begin{aligned}&(1+x)y' + y = \cos x,\\&[(1+x)y]' = \cos x,\\&(1+x)y = \int \cos x dx,\\&(1+x)y = \sin x + C.\end{aligned}$$

The initial condition implies that $C = 1$. Therefore, the solution to (1) is

$$y = \frac{1+\sin x}{1+x}.$$

2. Find the (implicit) solution of the exact differential equation

$$(x+y)y' = -(x+\arctan y)(1+y^2), \qquad y(0)=0. \tag{2}$$

A. $\frac{x^2}{2} + x\arctan y = 0$
B. $\frac{x^2}{2} + x\arctan y + \frac{1}{2}\ln(1+y^2) = 0$
C. $y = \tan\left(-\frac{x}{2}\right)$
D. $y^2 = e^{-x^2} - 1$
E. $y = e^{-x^2} - 1$
F. none of the above

B

Rewriting (2), we have

$$(x+\arctan y) + \frac{x+y}{1+y^2}y' = 0,$$

so we seek a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = x + \arctan y, \qquad \frac{\partial F}{\partial y} = \frac{x+y}{1+y^2}.$$

Integrating with respect to x, we have

$$F(x,y) = \frac{x^2}{2} + x\arctan y + g(y).$$

Integrating with respect to y, we have

$$F(x,y) = x\arctan y + \frac{1}{2}\ln(1+y^2) + h(x).$$

It follows that

$$F(x,y) = \frac{x^2}{2} + x\arctan y + \frac{1}{2}\ln(1+y^2).$$

Thus, we obtain the implicit solution

$$\frac{x^2}{2} + x\arctan y + \frac{1}{2}\ln(1+y^2) = C.$$

Imposing the initial condition $y(0)=0$, we have $C=0$.

3. Find the solution to the initial value problem:

$$2y''' - 3y'' - 2y' = 0, \qquad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 3. \tag{3}$$

A. $-\frac{7}{2} + 4e^{-\frac{1}{2}t} + \frac{1}{2}e^{2t}$

B. $1 - 2e^{-\frac{1}{2}t} + 2e^{2t}$

C. $\cosh t + \frac{1}{2}\sinh(2t)$

D. $\cos t + \frac{1}{2}\sin(2t)$

E. $e^{it} + 3e^{-it}$

F. none of the above

A

For (3), the characterisitic equation is

$$2r^3 - 3r^2 - 2r = 0,$$
$$r(2r+1)(r-2) = 0.$$

It follows that the general solution is

$$y(t) = c_1 + c_2 e^{-\frac{1}{2}t} + c_3 e^{2t},$$

and

$$y'(t) = -\frac{1}{2}c_2 e^{-\frac{1}{2}t} + 2c_3 e^{2t},$$
$$y''(t) = \frac{1}{4}c_2 e^{-\frac{1}{2}t} + 4c_3 e^{2t}.$$

The initial condition implies that

$$c_1 + c_2 + c_3 = 1,$$
$$-\frac{1}{2}c_2 + 2c_3 = -1,$$
$$\frac{1}{4}c_2 + 4c_3 = 3.$$

Thus, we have

$$c_1 = -\frac{7}{2}, \quad c_2 = 4, \quad c_3 = \frac{1}{2}.$$

4. Which of the following is NOT a solution of the following differential equation?

$$y'' + 4y = \cos 3t \tag{4}$$

A. $-\frac{1}{5}\cos 3t$
B. $\cos 2t - \frac{1}{5}\cos 3t$
C. $\sin 2t - \frac{1}{5}\cos 3t$
D. $e^{2ti} + e^{-2ti} - \frac{1}{5}\cos 3t$
E. $\cos 2t + \sin 2t$
F. all of the above are solutions of (4)

E

Using the method of undetermined coefficients or variation of parameters, we find that

$$Y(t) = -\frac{1}{5}\cos 3t$$

is a particular solution. Thus, the general solution is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{5}\cos 3t.$$

5. For the differential equation (5), what is the lower bound of the radius of convergence for the power series solution around 0?

$$(1+x^2)y'' - \cos x \cdot y' + e^x y = 0. \tag{5}$$

A. 0
B. 1
C. $\sqrt{2}$
D. π
E. ∞
F. none of the above

B

Rewriting (5), we have

$$y'' - \frac{\cos x}{1+x^2} \cdot y' + \frac{e^x}{1+x^2} y = 0.$$

The radius of convergence for the Taylor series around 0 of $\cos x$ and e^x is ∞, so we just need the denominator. If we set

$$x^2 + 1 = 0,$$

then we obtain complex roots $x = \pm i$. Thus, the radius of convergence of the Taylor series around 0 of $\frac{1}{1+x^2}$ is 1, and the result follows.

You may also obtain the result using the ratio test for the Taylor series:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots .$$

6. If

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

is the Taylor series around 0 for the function

$$\cos(1+x) + \sin(1+x),$$

then what is the value of a_{2015}?

A.
$$a_{2015} = \frac{\sin 1 - \cos 1}{2015}$$

B.
$$a_{2015} = \frac{\cos 1 - \sin 1}{2015!}$$

C.
$$a_{2015} = \frac{\sin 1 - \cos 1}{2015!}$$

D.
$$a_{2015} = \frac{\pi}{2015!}$$

E.
$$a_{2015} = \frac{\pi^{2015}}{2015!}$$

F. none of the above

C

We have

$$\begin{aligned}
\cos(1+x) + \sin(1+x) &= \cos 1 \cos x - \sin 1 \sin x + \sin 1 \cos x + \cos 1 \sin x \\
&= (\cos 1 + \sin 1) \cos x + (\cos 1 - \sin 1) \sin x \\
&= (\cos 1 + \sin 1) \left(1 - \frac{x^2}{2!} + \cdots\right) + (\cos 1 - \sin 1) \left(x - \frac{x^3}{3!} + \cdots\right).
\end{aligned}$$

It follows that

$$a_{2n} = \frac{(-1)^n (\cos 1 + \sin 1)}{(2n)!}, \qquad a_{2n+1} = \frac{(-1)^n (\cos 1 - \sin 1)}{(2n+1)!}$$

and

$$a_{2015} = -\frac{\cos 1 - \sin 1}{2015!} = \frac{\sin 1 - \cos 1}{2015!}$$

Alternatively, you may find a_{2015} by finding

$$y^{(2015)}(0) = \sin 1 - \cos 1.$$

7. Find the inverse Laplace transform of

$$F(s) = \frac{e^{2-2s}(2s^2 + \pi^2 - 1)}{(s-1)(s^2+\pi^2)}.$$

A.

$$f(t) = e^t + e^2\cos(\pi t) + \frac{e^2}{\pi}\sin(\pi t)$$

B.

$$f(t) = u_2(t)\left[e^t + e^2\cos(\pi t) + \frac{e^2}{\pi}\sin(\pi t)\right]$$

C.

$$f(t) = u_2(t)\left[e^{t-2} + e^2\cos(\pi t) + \frac{e^2}{\pi}\sin(\pi t)\right]$$

D.

$$f(t) = u_2(t)\left[e^t + e^2\cos(t-2) + \frac{e^2}{\pi}\sin(t-2)\right]$$

E.

$$f(t) = u_2(t)\left[e^{t-2} + e^2\cos(t-2) + e^2\sin(t-2)\right]$$

F. none of the above

B

We define

$$H(s) = \frac{2s^2 + \pi^2 - 1}{(s-1)(s^2+\pi^2)}.$$

To do partial fraction of H, we begin with

$$\begin{aligned}\frac{2s^2+\pi^2-1}{(s-1)(s^2+\pi^2)} &= \frac{a}{s-1} + \frac{bs+c}{s^2+\pi^2} \\ &= \frac{(a+b)s^2 + (-b+c)s + (a\pi^2 - c)}{(s-1)(s^2+\pi^2)},\end{aligned}$$

and then solving

$$a+b=2, \quad -b+c=0, \qquad a\pi^2 - c = \pi^2 - 1,$$

we get

$$a=b=c=1.$$

The inverse Laplace transform of

$$H(s) = \frac{2s^2+\pi^2-1}{(s-1)(s^2+\pi^2)} = \frac{1}{s-1} + \frac{s}{s^2+\pi^2} + \frac{1}{\pi}\cdot\frac{\pi}{s^2+\pi^2}$$

is

$$h(t) = e^t + \cos(\pi t) + \frac{1}{\pi}\sin(\pi t).$$

Since

$$F(s) = e^2 e^{-2s} H(s),$$

we get

$$\begin{aligned} f(t) &= e^2 u_2(t) h(t-2) \\ &= u_2(t)\left[e^t + e^2\cos(\pi(t-2)) + \frac{e^2}{\pi}\sin(\pi(t-2))\right] \\ &= u_2(t)\left[e^t + e^2\cos(\pi t) + \frac{e^2}{\pi}\sin(\pi t)\right]. \end{aligned}$$

8. Find the Laplace transform of the piecewise continuous function

$$f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 2-t & \text{if } 1 \le t < 2 \\ 0 & \text{if } t \ge 2 \end{cases} \tag{6}$$

A.

$$F(s) = \frac{1}{s^2} - \frac{2}{s^2} + \frac{1}{s^2}$$

B.

$$F(s) = 1 - e^{-s} + e^{-2s}$$

C.

$$F(s) = 1 - 2e^{-s} + e^{-2s}$$

D.

$$F(s) = \frac{1}{s^2} + \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$$

E.

$$F(s) = \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$$

F. none of the above

E

Rewriting (6), we get

$$f(t) = t - 2u_1(t)(t-1) + u_2(t)(t-2).$$

It follows that

$$F(s) = \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}.$$

9. If $y(t)$ is the solution of the initial value problem:

$$y'' + 4y = \delta(t - 2014\pi), \qquad y(0) = 0, \quad y'(0) = 0, \tag{7}$$

then what is the value of $y(2015\pi)$?

A. 0
B. -1
C. 1
D. π
E. $\frac{\pi}{2}$
F. none of the above

A

Taking Laplace transform of (7), we get

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= e^{-2014\pi s}, \\ Y(s) &= \frac{e^{-2014\pi s}}{2} \cdot \frac{2}{s^2+4}. \end{aligned}$$

Thus, we obtain the unique solution

$$y(t) = \frac{1}{2} u_{2014\pi}(t) \sin[2(t - 2014\pi)],$$

and

$$y(2015\pi) = \frac{1}{2}\sin(2\pi) = 0.$$

10. Find the inverse Laplace transform of

$$F(s) = \ln\left(\frac{s^2+1}{s^2+4}\right). \tag{8}$$

A.

$$f(t) = e^{t^2+1} + e^{t^2+4}$$

B.

$$f(t) = e^{2t} + e^{-2t}$$

C.

$$f(t) = u_1(t)\cos t + u_2(t)\sin(2t)$$

D.

$$f(t) = \frac{2}{t}(2\cos t + 1)(\cos t - 1)$$

E.

$$f(t) = \frac{2}{t}(\cos t - \cos(2t))$$

F. none of the above

D

We use the property

$$\mathcal{L}\{-tf(t)\} = F'(s).$$

Rewriting (8), we have

$$F(s) = \ln(s^2+1) - \ln(s^2+4).$$

It follows that

$$F'(s) = \frac{2s}{s^2+1} - \frac{2s}{s^2+4},$$

and

$$\begin{aligned}-tf(t) &= \mathcal{L}^{-1}\{F'(s)\}\\ &= 2\cos t - 2\cos(2t)\\ &= 2\cos t - 4\cos^2 t + 2\\ &= 2(2\cos + 1)(-\cos t + 1).\end{aligned}$$

Hence,

$$f(t) = \frac{2}{t}(2\cos + 1)(\cos t - 1).$$

Note that E is off by a sign.

Part II. True/False $5 \times 2 = 10$ points

Choose A if the statement is true; choose B if the statement is false.

11. There is a unique solution to the initial value problem:

$$y' - y^{\frac{1}{2}} = 0, \qquad y(0) = 0.$$

12. If p and q are continuous functions, and y_1 and y_2 are solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

then we have

$$W(y_1, y_2)(t) = e^{-\int_{t_0}^{t} p(\tau)d\tau}.$$

for some constant t_0.

13. If p and q are continuous real-valued functions, and $u(t)+iv(t)$ are complex-valued solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0, \tag{9}$$

then both u and v are also solutions of (9).

14. For the power series

$$\sum_{n=0}^{\infty} a_n x^n,$$

if the radius of convergence is ρ and $|x_1| \geq \rho$, then the series

$$\sum_{n=0}^{\infty} a_n (x_1)^n$$

does not converge.

15. If f is continuous function on $[0, \infty)$, then the Laplace transform

$$\mathcal{L}\{f(t)\} = F(s)$$

exists for $s > 0$.

B B A B B

11. For example, both $y = \frac{t^2}{4}$ and $y = 0$ are solutions.
12. The original statement o Abel's theorem is that

$$W(y_1, y_2)(t) = c_0 e^{-\int_{t_0}^{t} p(\tau)d\tau},$$

where the constant

$$c_0 = W(y_1, y_2)(t_0)$$

could be zero. As stated,

$$W(y_1, y_2)(t) = e^{-\int_{t_0}^{t} p(\tau)d\tau}$$

is never zero. If $y_1 = y_2 = 0$, then $W(y_1, y_2)(t) = 0$, which contradicts the statement.

13. See Theorem 3.2.6

14. Compare to Problem 15. in Exam 3, we have $|x_1| \geq \rho$ instead of $|x_1| > \rho$. For example, for the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n} x^n,$$

the radius of convergence is 1. For $|x_1| > 1$, the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n} (x_1)^n$$

does not converge. However, by alternating test, you can check that the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

DOES converge.

15. See Theorem 6.1.2. For an counter-example, let $f(t) = e^t$, then

$$\mathcal{L}\{e^t\} = \frac{1}{s-1}$$

does not exists for $0 \leq s \leq 1$.

Math 217 Final Exam Dec 11, 2015

Part III will be collected separately. Please write your NAME and your STUDENT NUMBER.

Student Number:

Name:

For graders:

16a + 16b. + 16c. + 16d.

16e.

17.

Total:

Part III. Hand-graded problems $\quad 10 + 10 + 20 = 40$ points

16a. (4 points)
Find the general solution of the homogenous differential equation:

$$y'' - 4y = 0. \tag{10}$$

The characteristic equation of (10) is

$$r^2 - 4 = 0$$

with roots $r_1 = 2$ and $r_2 = -2$. Thus, $y_1(t) = e^{2t}$ and $y_2 = e^{-2t}$ the general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{2t} + c_2 e^{-2t}.$$

16b. (6 points) Use the method of undetermined coefficients to find the general solution of the non-homogeneous differential equation:

$$y'' - 4y = \sinh 2t. \tag{11}$$

We try a particular solution

$$Y(t) = Ate^{2t} + Bte^{-2t}.$$

It follows that

$$Y''(t) = 4Ae^{2t} - 4Be^{-2t} + 4Ate^{2t} + 4Bte^{-2t}.$$

Thus, (11) implies that

$$4Ae^{2t} - 4Be^{-2t} = \frac{e^{2t} - e^{-2t}}{2},$$

and so $A = B = \frac{1}{8}$. Hence, the general solution of (11) is

$$\begin{aligned} y(t) &= c_1 e^{2t} + c_2 e^{-2t} + \frac{te^{2t}}{8} + \frac{te^{-2t}}{8} \\ &= c_1 e^{2t} + c_2 e^{-2t} + \frac{t\cosh(2t)}{4}. \end{aligned}$$

This page is intentionally left blank.

16c. (6 points)

Use the method of variation of parameters to find the general solution of the non-homogeneous differential equation:

$$y'' - 4y = \sinh 2t. \tag{12}$$

Since $y_1(t) = e^{2t}$ and $y_2 = e^{-2t}$, it follows that

$$W(y_1, y_2)(t) = -4.$$

Hence, the general solution is

$$\begin{aligned}
y(t) &= -y_1(t) \int \frac{y_2(t)\sinh(2t)}{W(y_1,y_2)(t)} dt + y_2(t) \int \frac{y_1(t)\sinh(2t)}{W(y_1,y_2)(t)} dt \\
&= e^{2t} \int \frac{1}{8}\left(1 - e^{-4t}\right) dt + e^{-2t} \int -\frac{1}{8}\left(e^{4t} - 1\right) dt \\
&= e^{2t} \left[\frac{t}{8} + \frac{e^{-4t}}{32} + c_1 + \frac{1}{32}\right] + e^{-2t} \left[-\frac{e^{4t}}{32} + \frac{e^{t}}{8} + c_2 - \frac{1}{32}\right] \\
&= c_1 e^{2t} + c_2 e^{-2t} + \frac{t\cosh(2t)}{4}.
\end{aligned}$$

Note that for convenience, we have use the integration constants $c_1 + \frac{1}{32}$ and $c_2 - \frac{1}{32}$.

This page is intentionally left blank.

16d. (4 points)
Use the results in 16b. or 16c. to find the unique solution of the initial value problem:

$$y'' - 4y = \sinh 2t, \qquad y(0) = 0, \quad y'(0) = 0. \tag{13}$$

The general solution of (11) is

$$y(t) = c_1 e^{2t} + c_2 e^{-2t} + \frac{t\cosh(2t)}{4}.$$

Taking direvative, we get

$$y(t) = 2c_1 e^{2t} - 2c_2 e^{-2t} + \frac{\cosh(2t)}{4} + \frac{t\sinh(2t)}{4}.$$

The initial condition implies that

$$c_1 + c_2 = 0, \qquad 2c_1 - 2c_2 + \frac{1}{4} = 0.$$

It follows that

$$c_1 = -\frac{1}{16}, \qquad c_2 = \frac{1}{16}.$$

Hence, the unique solution of (13) is

$$\begin{aligned} y(t) &= -\frac{1}{16}e^{2t} + \frac{1}{16}e^{-2t} + \frac{t\cosh(2t)}{4} \\ &= -\frac{\sinh(2t)}{8} + \frac{t\cosh(2t)}{4}. \end{aligned}$$

16e. (10 points)
Use the method of Laplace transform to find the solution of the initial value problem:

$$y'' - 4y = \sinh 2t, \qquad y(0) = 0, \quad y'(0) = 0. \tag{14}$$

Taking Laplace transform of (14), we get

$$\begin{aligned}
s^2 Y(s) - sy(0) - y'(0) - 4Y(s) &= \frac{2}{s^2 - 4}, \\
Y(s) = \frac{2}{(s^2-4)^2} &= \frac{1}{2} \cdot \frac{2}{s^2-4} \cdot \frac{2}{s^2-4}.
\end{aligned}$$

Using the convolution integral, we get

$$\begin{aligned}
y(t) &= \frac{1}{2} \int_0^t \sinh[2(t-\tau)] \sinh(2\tau) d\tau \\
&= \frac{1}{8} \int_0^t \left(e^{2t-2\tau} - e^{2\tau-2t}\right)\left(e^{2\tau} - e^{-2\tau}\right) d\tau \\
&= \frac{1}{8} \int_0^t \left(e^{2t} + e^{-2t} - e^{4\tau-2t} - e^{2t-4\tau}\right) d\tau \\
&= \frac{t\left(e^{2t} + e^{-2t}\right)}{8} - \frac{1}{4} \int_0^t \cosh(4\tau - 2t) d\tau \\
&= \frac{t\cosh(2t)}{4} - \frac{1}{16} \sinh(4\tau - 2t)\Big|_0^t \\
&= \frac{t\cosh(2t)}{4} - \frac{1}{16} \sinh(2t) + \frac{1}{16} \sinh(-2t) \\
&= \frac{t\cosh(2t)}{4} - \frac{\sinh(2t)}{8}.
\end{aligned}$$

This page is intentionally left blank.

17. Find the solution of the initial value problem:

$$y'' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 1 \tag{15}$$

where

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < \pi \\ 0 & \text{if } t \ge \pi \end{cases} \tag{16}$$

Using (16) to rewriting f, we have

$$f(t) = 1 - u_\pi(t).$$

Taking the Laplace transform of (15), we get

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + Y(s) &= \frac{1}{s} - \frac{e^{-\pi s}}{s}, \\ s^2Y(s) - 1 + Y(s) &= \frac{1}{s} - \frac{e^{-\pi s}}{s}. \end{aligned}$$

It follows that

$$\begin{aligned} Y(s) &= \frac{1}{s(s^2+1)} + \frac{e^{-\pi s}}{s(s^2+1)} + \frac{1}{s^2+1} \\ &= \frac{1}{s} - \frac{s}{s^2+1} + \frac{1}{s^2+1} - e^{-\pi s} \cdot \frac{1}{s} + e^{-\pi s} \cdot \frac{s}{s^2+1}. \end{aligned}$$

Taking the inverse Laplace transform, we obtain the solution of (15)

$$\begin{aligned} y(t) &= 1 - \cos t + \sin t - u_\pi(t)\left[1 - \cos(t - \pi)\right] \\ &= 1 - \cos t + \sin t - u_\pi(t)(1 + \cos t). \end{aligned}$$

This page is intentionally left blank.

This page is intentionally left blank.