Lie groupoids and Lie algebroids

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Lie groupoid

Lie groupoid

A Lie groupoid $\mathcal{G}$ over the base manifold M is a category such that

- the set of objects is M, and the set of arrows $\mathcal{G}$ is a manifold;
- the arrows are invertible;
- the source s and target t are submersions;
- the multiplication m and the identity id are smooth.

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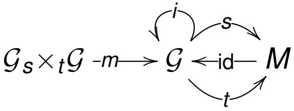
The structure maps are summarized by the following commutative diagram:



Source-simply-connected Lie groupoid

Notation

For a Lie groupoid $\mathcal{G}$ over M, we denote it by $\mathcal{G} \rightrightarrows M$.

Source-simply-connected Lie groupoid

For a Lie groupoid $\mathcal{G} \rightrightarrows M$,

- for $x \in M$, the source fiber of x is $s^{-1}(x)$, and the target fiber is $t^{-1}(x)$;
- $\mathcal{G} \rightrightarrows M$ is source-connected, if $s^{-1}(x)$ is connected for each $x \in M$;
- $\mathcal{G} \rightrightarrows M$ is source-simply-connected, if $s^{-1}(x)$ is connected and simply-connected for each $x \in M$.

Examples: Lie groupoids

1. Lie group

If the base M is a point, then $\mathcal{G}$ is a Lie group.

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2. Bundle of Lie groups

If the source $s : \mathcal{G} \to M$ and the target $t : \mathcal{G} \to M$ coincide, then $\mathcal{G} \rightrightarrows M$ is a bundle of Lie groups.

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3. Pair groupoid

For a connected manifold M, we define the ***pair groupoid*** $\mathrm{Pair}(M) \rightrightarrows M$:

- $\mathrm{Pair}(M) = M \times M$;
- $s : \mathrm{Pair}(M) \to M, \quad (x, y) \mapsto x$;
- $t : \mathrm{Pair}(M) \to M, \quad (x, y) \mapsto y$;
- $\mathrm{id} : M \to \mathrm{Pair}(M), \quad x \mapsto (x, x)$;
- $m : \mathrm{Pair}(M)_s \times_t \mathrm{Pair}(M) \to \mathrm{Pair}(M), \quad ((x, y), (y, z)) \to (x, z)$.

Examples: Lie groupoids

4. Fundamental groupoid

For a path-connected manifold M, we define the ***fundamental groupoid*** $\Pi_1(M) \rightrightarrows M$:

- $\Pi_1(M) = \{[\gamma] \mid \gamma : I \to M\}$ is the homotopy classes of paths;
- $s : \Pi_1(M) \to M, \quad [\gamma] \mapsto \gamma(0)$;
- $t : \Pi_1(M) \to M, \quad [\gamma] \mapsto \gamma(1)$;
- $\mathrm{id} : M \to \Pi_1(M), \quad x \mapsto \mathrm{id}(x)$
  where $\mathrm{id}(x)$ is the constant path at x;
- $m : \Pi_1(M)_s\times_t\Pi_1(M) \to \Pi_1(M)$ is the concatenation of paths.

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Remark

Note that $(s, t) : \Pi_1(M) \to \mathrm{Pair}(M)$ is a Lie groupoid morphism.

Examples: Lie groupoids

5. General linear groupoid

For a vector bundle $E \to M$, we define the ***general linear groupoid*** $\mathtt{GL}(E) \rightrightarrows M$:

- $\mathtt{GL}(E) \simeq \{(x, y, \phi_{xy}) \mid x \in M,\ y \in M\}$ where $\phi_{xy} : E_x \xrightarrow{\sim} E_y$ is an isomorphism of vector spaces;
- $s : \mathtt{GL}(E) \to M, \quad (x, y, \phi_{xy}) \mapsto x;$
- $t : \mathtt{GL}(E) \to M, \quad (x, y, \phi_{xy}) \mapsto y;$
- $\mathrm{id} : M \to \mathtt{GL}(E), \quad x \mapsto (x, x, \mathrm{id}_{xx});$
- $m : \mathtt{GL}(E)_s\times_t\mathtt{GL}(E) \to \mathtt{GL}(E)$, $\big((x, y, \phi_{xy}), (y, z, \phi_{yz})\big) \mapsto (x, z, \phi_{yz} \circ \phi_{xy})$.

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Remark

A Lie groupoid action of $\mathcal{G} \rightrightarrows M$ on a vector bundle $E \to M$ is a Lie groupoid morphism $\rho : \mathcal{G} \to \mathtt{GL}(E)$.

Examples: Lie groupoids

6. Holonomy groupoid

For a vector bundle $E \to M$, we define the ***holonomy groupoid*** $\mathtt{Hol}(E) \rightrightarrows M$:

- $\mathtt{Hol}(E) = \{([\gamma], \phi_{st}) \mid [\gamma] \in \Pi_1 D$.
  where $\phi_{st} : E_{\gamma(0)} \xrightarrow{\sim} E_{\gamma(1)}\}$ is an isomorphism of vector spaces;
- $s : \mathtt{Hol}(E) \to M, \quad ([\gamma], \phi_{st}) \mapsto \gamma(0);$
- $t : \mathtt{Hol}(E) \to M, \quad ([\gamma], \phi_{st}) \mapsto \gamma(1);$
- $\mathrm{id} : M \to \mathtt{Hol}(E), \quad x \mapsto (\mathrm{id}(x), \mathrm{id}_{xx});$
- $m : \mathtt{Hol}(E)_s \times_t \mathtt{Hol}(E) \to \mathtt{Hol}(E),$
  $((\gamma, \phi_{st}), (\sigma, \psi_{st})) \mapsto (\gamma \circ \sigma, \phi_{st} \circ \psi_{st}).$

Examples: Lie groupoids

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Remark

There is an obvious surjective groupoid morphism $\mathtt{Hol}(E) \to \mathtt{GL}(E)$ which covers $\Pi_1(M) \to \mathrm{Pair}(M)$.

Examples: Lie groupoids

7. Gauge groupoid

For a principal G-bundle $\pi : P \to M$, we consider the diagonal action of G on $P \times P$:

$$g(u, v) = (gu,\ gv).$$

We define the ***gauge groupoid*** $\mathtt{Gg}(P) \rightrightarrows M$:

- $\mathtt{Gg}(P) = (P \times P)/G$.

For $u, v, v', w \in P$ such that $\pi(v) = \pi(v') = x$,

- $s([u, v]) = \pi(u)$;
- $t([u, v]) = \pi(v)$;
- $\mathrm{id}(x) = ([v, v])$;
- $m\,([u, v], [v', w]) = ([u, w])$.

Lie algebroid

Lie algebroid

A Lie algebroid A over the base manifold M is a vector bundle $A \to M$ with a Lie bracket $[\cdot, \cdot]$ on $\Gamma(A)$ and an anchor map $a : A \to TM$ that preserves the bracket and satisfies the Leibniz rule

$$[X, fY] = f[X, Y] + a(X)(f)Y. \tag{2.1}$$

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Lie functor

For a Lie groupoid $\mathcal{G} \rightrightarrows M$, the vector bundle

$$\mathtt{Lie}(\mathcal{G}) \doteq \ker\left(Ts : T\mathcal{G} \to TM\right)|_{\mathrm{id}(M)} \tag{2.2}$$

with the bracket on left invariant vector fields, and the anchor $Tt : \mathtt{Lie}(\mathcal{G}) \to TM$, is Lie algebroid.

Examples: Lie algebroids

1.Lie algebra

If the base M is a point, then A is a Lie algebra.

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2. Bundle of Lie algebras

If the anchor $a : A \to TM$ is trivial, then A is a bundle of Lie algebras.

Examples: Lie algebroids

1.Lie algebra

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2. Bundle of Lie algebras

If the anchor $a : A \to TM$ is trivial, then A is a bundle of Lie algebras.

3. Tangent algebroid

- For a manifold M, the tangent bundle TM itself is a Lie algebroid.
- Both the pair groupoid $\mathrm{Pair}(M)$ and the fundamental groupoid $\Pi_1(M)$ integrates the tangent algebroid TM. That is,

$$\mathtt{Lie}(\mathrm{Pair}(M)) = \mathtt{Lie}(\Pi_1(M)) = TM.$$

Examples: Lie algebroids

4. General linear algebroid

For a vector bundle $\pi : E \to M$ of rank r, we define the ***general linear algebroid*** $\mathfrak{gl}(E)$:

- Let $\mathbb{E}$ be the Euler vector field, we have

$$\Gamma(\mathfrak{gl}(E)) = \{X \in \Gamma(TE) \mid \mathcal{L}_{\mathbb{E}} X = 0\}. \tag{2.3}$$

- The anchor $a : \mathfrak{gl}(E) \to TM$ is defined by $\pi_* : TE \to TM$. A section $X \in \Gamma(\mathfrak{gl}(E))$ is invariant under the fiber rescaling of $E \to M$, so

$$a(X) = \pi_*(X) \in \Gamma(TM)$$

  is well-defined.
- The sections $\Gamma(\mathfrak{gl}(E))$ are the derivations of E.

Examples: Lie algebroids

Remark

- The general linear algebroid $\mathfrak{gl}(E)$ fits into the following exact sequence
$$0 \longrightarrow V \longrightarrow \mathfrak{gl}(E) \xrightarrow{a} TM \longrightarrow 0 \tag{2.4}$$
where V is the vector bundle whose sections are the vertical vector fields, i.e. vector fields on E that are tangent to the fibered of $E \to M$.
- Both the holonomy groupoid $\mathtt{Hol}(E)$ and the general linear groupoid $\mathtt{GL}(E)$ integrate the general linear algebroid $\mathfrak{gl}(E)$.
- A Lie algebroid action $A \to M$ on a vector bundle $E \to M$ is a Lie algebroid morphism from A to $\mathfrak{gl}(E)$.

Examples: Lie algebroids

5. Atiyah algebroid

For a principal G-bundle $P \to M$, the ***Atiyah algebroid*** $\mathtt{At}(P)$ of P is the Lie algebroid of the gauge groupoid $\mathtt{Gg}(P) \rightrightarrows P$, which fits into the following short exact sequence:

$$0 \longrightarrow P \times_G \mathfrak{g} \longrightarrow \mathtt{At}(P) \longrightarrow TM \longrightarrow 0 \qquad (2.5)$$

where $P \times_G \mathfrak{g}$ is the associated bundle.

Integrations of a Lie algebroid

- In general, it is not always true that a Lie algebroid integrates to a Lie groupoid. The integrability condition was given in [Crainic-Fernandes, Ann. Math. 2003].
- For an integrable Lie algebroid A, there is a unique, up to isomorphism, source-simply-connected groupoid $\mathcal{G}^{ssc}$ integrating A.
- For another Lie groupoid $\mathcal{G} \rightrightarrows M$ integrating A, there is a groupoid morphism $\mathcal{G}^{ssc} \to \mathcal{G}$.
- In some cases, there also exists an adjoint groupoid $\mathcal{G}^{adj} \rightrightarrows M$ that receives a map from other integration of A.

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- In some cases, there also exists an adjoint groupoid $\mathcal{G}^{adj} \rightrightarrows M$ that receives a map from other integration of A.

1. Lie groups

For a semi-simple Lie algebra $\mathfrak{g}$, we have the simply-connected Lie group G^{sc}, and the adjoint Lie groupd G^{adj}.

Examples of integrations

2. Tangent integrations

- For a path-connected manifold M, the integrations of the tangent algebroid TM is given by normal subgroups of the fundamental group, $\Lambda(\pi_1(M, x))$.
- The equivalence is given by

$$\mathcal{G} \rightrightarrows M \quad \mapsto \quad t_*[\pi_1(s^{-1}(x), \mathrm{id}(x))]. \tag{3.1}$$

- The fundamental groupoid $\Pi_1(M)$ is the source-simply-connected integration, and corresponds to the trivial group $\{1\} < \pi_1(M, x)$.
- The pair groupoid $\mathrm{Pair}(M)$ is the adjoint integration, and corresponds to $\pi_1(M, x)$.

Examples of integrations

3. Log tangent algebroid

Let L be a closed hypersurface of M. The log tangent algebroid

$$T(M, \log L)$$

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is a Lie algebroid whose sections are the vector fields tangent to L.

- In this case, there exists an adjoint groupoid integrating $T(M, \log L)$, which we call the log pair groupoid.

Examples of integrations

- Take the pair groupoid $\mathrm{Pair}(M) = M \times M$, and the subgroupoid $\mathrm{Pair}(L) = L \times L$. We blow up $\mathrm{Pair}(M)$ along $\mathrm{Pair}(L)$ and obtain the blow-down map
$$p : \mathrm{Bl}_{\mathrm{Pair}(L)}(\mathrm{Pair}(M)) \to \mathrm{Pair}(M).$$
Note: this is the real projective blow-up.
- From $\mathrm{Bl}_{\mathrm{Pair}(L)}(\mathrm{Pair}(M))$, we remove the closures of $p^{-1}(L \times M)$ and $p^{-1}(M \times L)$, i.e. the proper transforms of $s^{-1}(L)$ and $t^{-1}(L)$.
- It turns out that the log pair groupoid is
$$[\mathrm{Pair}(M) : \mathrm{Pair}(L)] = \mathrm{Bl}_{\mathrm{Pair}(L)}(\mathrm{Pair}(M)) \setminus (\overline{p^{-1}(L \times M)} \cup \overline{p^{-1}(M \times L)})$$
and the blow-down map
$$p : [\mathrm{Pair}(M) : \mathrm{Pair}(L)] \to \mathrm{Pair}(M)$$
is a Lie groupoid morphism.

Examples of integrations

4. Elementary modification

Generalizing the log tangent algebroid, if A is a Lie algebroid over M, and B is a Lie subalgebroid over a hypersurface $L \subset M$, then we define the Lie algebroid $[A:B]$ with sheaf of sections

$$[A:B](U) = \{X \in \Gamma(U, A) \mid X|_L \in \Gamma(U \cap L, B)\}.$$

That is, a section X of $[A:B]$ is a section of A such that when $X|_L$ is a section of B.

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Generalizing the log tangent algebroid, if A is a Lie algebroid over M, and B is a Lie subalgebroid over a hypersurface $L \subset M$, then we define the Lie algebroid $[A\colon B]$ with sheaf of sections

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That is, a section X of $[A\colon B]$ is a section of A such that when $X|_L$ is a section of B.

- If $\mathcal{G} \rightrightarrows M$ integrates A and $\mathcal{H} \rightrightarrows L$ is a subgroupoid of $\mathcal{G}$ integrating B, then we may blow up $\mathcal{G}$ along $\mathcal{H}$, and remove the proper transforms of $s^{-1}(L)$ and $t^{-1}(L)$ as before:

$$[\mathcal{G}\colon\mathcal{H}] = \mathrm{Bl}_{\mathcal{H}}(\mathcal{G}) \setminus (\overline{p^{-1}(s{-}1(L))} \cup \overline{p^{-1}(t^{-1}(L))}).$$

Examples of integrations

- it turns out that $[\mathcal{G} : \mathcal{H}]$ is a Lie groupoid integrating $[A : B]$ and the blow-down map
$$p : [\mathcal{G} : \mathcal{H}] \to \mathrm{Pair}(M)$$
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Thank you!