Ma 233: Calculus III
Solutions to Midterm Examination 1

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18 questions on 9 pages

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1. Find the distance between the point $(1, -2, 3)$ and the yz plane.

   (a) $1 \Leftarrow$

   (b) $\sqrt{2}$

   (c) $\sqrt{3}$

   (d) 2

   (e) $\sqrt{5}$

   (f) $\sqrt{6}$

   (g) $\sqrt{7}$

   (h) $\sqrt{8}$

   (i) 3

   Solution: This is the distance between the origin and $(1, 0, 0)$, or 1. $\square$

2. Find the radius of the sphere $S = \{(x, y, z) : x^2 + y^2 + z^2 + 2x - 3y + 4z = 5\}$

   (a) $\frac{1}{2}$

   (b) $\frac{\sqrt{2}}{2}$

   (c) $\frac{\sqrt{3}}{2}$

   (d) $\frac{\sqrt{5}}{2}$

   (e) $\frac{\sqrt{7}}{2}$

   (f) $\frac{3}{2}$

   (g) $\frac{5}{2}$

   (h) $\frac{7}{2} \Leftarrow$

   (i) $\frac{9}{2}$

   Solution: Complete the squares: $x^2 + y^2 + z^2 + 2x - 3y + 4z = (x + 1)^2 - 1 + (y - \frac{3}{2})^2 - \frac{9}{4} + (z + 2)^2 - 4 = 5$, so the standard form is

   $$(x - [-1])^2 + (y - \frac{3}{2})^2 + (z - [-2])^2 = \frac{49}{4}.$$

   Read off the radius $\sqrt{\frac{49}{4}} = \frac{7}{2}$. $\square$

3. Let S be the set of all points at a distance of 5 from the origin that lie in a plane perpendicular to the z-axis containing the point $P = (1, 2, 3)$. Which of the following points are contained in S:

(I) $(4, 0, 3)$

(II) $(0, 4, 3)$

(III) $(0, 0, 3)$

(IV) $(2, 2, 3)$

(a) I only.
(b) II only.
(c) III only.
(d) IV only.
(e) I and II only. $\Leftarrow$
(f) II and III only.
(g) III and IV only.
(h) I and IV only.
(i) All.
(j) None.

Solution: Set S is a circle of radius $4 = \sqrt{5^2 - 3^2}$ centered at $(0, 0, 3)$, lying in the plane $\{(x, y, z) : z = 3\}$. The points $(0, 0, 3)$ and $(2, 2, 3)$ are not contained in S, as their distances from $(0, 0, 3)$ are 0 and $\sqrt{8}$, respectively. □

4. Let $A = (3, -5, 1)$ and $B = (2, 4, -6)$ be two points. Find the components of the vector $\vec{AB}$.

(a) $\langle -1, 9, -7 \rangle$ $\Leftarrow$
(b) $\langle 1, -9, 7 \rangle$
(c) $\langle 1, 9, 7 \rangle$
(d) $\langle -1, -9, -7 \rangle$
(e) $\langle 1, 9, -7 \rangle$
(f) $\langle -1, -9, 7 \rangle$
(g) $\langle 1, 9, -7 \rangle$
(h) $\langle -1, -9, 7 \rangle$
(i) $\langle 5, -1, -5 \rangle$
(j) $\langle -5, 1, 5 \rangle$

Solution: Components of $\vec{AB}$ are the coordinates of $B - A = (-1, 9, -7)$. □

5. Let $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{j} - 2\mathbf{k}$. Find $|2\mathbf{b} - \mathbf{a}|$.

   (a) $\sqrt{5}$
   (b) $\sqrt{10}$
   (c) $\sqrt{15}$
   (d) $\sqrt{20}$
   (e) 5
   (f) $\sqrt{30}$
   (g) $\sqrt{35}$ $\Leftarrow$
   (h) $\sqrt{40}$
   (i) $\sqrt{45}$
   (j) $\sqrt{50}$

   Solution: $2\mathbf{b} - \mathbf{a} = -\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$, so $|2\mathbf{b} - \mathbf{a}| = \sqrt{(-1)^2 + 3^2 + (-5)^2} = \sqrt{35}$. $\square$

6. Find a unit vector parallel to $\mathbf{a} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

   (a) $\langle 2, -1, -2 \rangle$
   (b) $\langle -2, 1, 2 \rangle$
   (c) $\langle 1, 1, 1 \rangle$
   (d) $\langle 1, -1, -1 \rangle$
   (e) $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$
   (f) $\left\langle \frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}} \right\rangle$
   (g) $\left\langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \right\rangle$ $\Leftarrow$
   (h) $\left\langle -\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \right\rangle$
   (i) $\left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle$
   (j) $\left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle$

   Solution: $|\mathbf{a}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = 3$, so $\mathbf{a} = |\mathbf{a}|(\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k})$. $\square$

7. Find the volume of the parallelopiped whose sides are given by $\vec{AB} = \langle 1, 1, 1 \rangle$, $\vec{AC} = \langle 1, -1, 1 \rangle$, and $\vec{AD} = \langle 1, -1, -1 \rangle$.

   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) 4 $\Leftarrow$
   (f) 5
   (g) 6
   (h) 7
   (i) 8
   (j) 9

   Solution: Compute the scalar triple product:

   $$\begin{aligned} \vec{AB} \cdot (\vec{AC} \times \vec{AD}) &= \langle 1, 1, 1 \rangle \cdot \langle 2, 2, 0 \rangle \\ &= 2 + 2 + 0 = 4. \end{aligned}$$

   The absolute value of this triple product is the desired volume. □

8. Find the scalar projection of vector $\mathbf{b} = \langle 1, -1, 2 \rangle$ onto $\mathbf{a} = \langle 1, 2, 2 \rangle$.

   (a) -4
   (b) -3
   (c) -2
   (d) -1
   (e) 0
   (f) 1 $\Leftarrow$
   (g) 2
   (h) 3
   (i) 4
   (j) 5

   Solution: Use the formula

   $$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

   with $\mathbf{a} \cdot \mathbf{b} = (1)(1) + (2)(-1) + (2)(2) = 3$ and $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 2^2} = 3$ to get $\text{comp}_{\mathbf{a}}\mathbf{b} = 1$. □

9. Find the area of the triangle with vertices $(3,2)$, $(8,6)$, and $(13,9)$.

   (a) 0.5
   (b) 1
   (c) 1.5
   (d) 2
   (e) 2.5 $\Leftarrow$
   (f) 3
   (g) 3.5
   (h) 4
   (i) 4.5
   (j) 5

   Solution: Write $A = (3,2)$, $B = (8,6)$, and $C = (13,9)$. The triangle is half the parallelogram determined by the vectors $\vec{AB} = \langle 5,4\rangle$ and $\vec{AC} = \langle 10,7\rangle$, so use half the cross-product formula for the area of the parallelogram:

$$\frac{|\langle 5,4,0\rangle \times \langle 10,7,0\rangle|}{2} = \frac{|\langle 0,0,-5\rangle|}{2} = 2.5$$

□

10. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be vectors. Which of the following expressions are meaningful:

    (I) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

    (II) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{a} \cdot \mathbf{d})$

    (III) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$

    (IV) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

    (a) I only.
    (b) II only.
    (c) III only.
    (d) IV only.
    (e) I and II only.
    (f) II and III only.
    (g) III and IV only.
    (h) I and IV only. $\Leftarrow$
    (i) All.
    (j) None.

    Solution: II and II are not meaningful because the factors in parentheses are scalars when they must be vectors to participate in a cross product. □

11. A worker pushes a box up a ramp that is inclined at an angle of 45 degrees above the horizontal. While doing it he exerts a force of 10 Newtons directed at an angle of 15 degrees above the horizontal. Find the distance in meters traveled by the box if the work done on it is 15 Newton-meters.

    (a) $3/\sqrt{2}$
    (b) $1/\sqrt{3}$
    (c) $\sqrt{3}/2$
    (d) $3/4$
    (e) 1
    (f) $\sqrt{3} \Leftarrow$
    (g) $\sqrt{2}/3$
    (h) $2/\sqrt{3}$
    (i) $4/3$
    (j) 2

    Solution: Denote the work done by w, so $w = 15Nm$. But $w = \mathbf{F} \cdot \mathbf{d} = |\mathbf{F}||\mathbf{d}| \cos\theta$, where θ is the angle between the force vector $\mathbf{F}$ and the displacement vector $\mathbf{d}$. Given $|\mathbf{F}| = 10N$ and $\theta = 45° - 15° = 30°$, so $\cos\theta = \sqrt{3}/2$, compute

$$|\mathbf{d}| = \frac{w}{|\mathbf{F}|\cos\theta} = \frac{15}{10(\sqrt{3}/2)} = \sqrt{3}.$$

□

12. Find two unit vectors orthogonal to both $\mathbf{u} = \langle 1, -2, 1\rangle$ and $\mathbf{v} = \langle -2, 4, -2\rangle$.

    (a) $\langle 1, 1, 1\rangle$ and $\langle 0, 0, 0\rangle$
    (b) $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$ and $\langle 1, 0, 0\rangle$
    (c) $\langle 1, 1, 1\rangle$ and $\langle -1, -1, -1\rangle$
    (d) $\langle 0, 1, 0\rangle$ and $\langle 0, -1, 0\rangle$
    (e) $\left\langle 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$ and $\left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right\rangle \Leftarrow$
    (f) $\left\langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$ and $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$
    (g) $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{j}$
    (h) $\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$
    (i) $\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$ and $\frac{2}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$
    (j) There is no solution

    Solution: Note first that $\mathbf{u}$ is (anti)parallel to $\mathbf{v} = -2\mathbf{u}$. Hence it suffices to find any two unit vectors orthogonal to $\mathbf{u}$. Of those given, only the pair $\left\langle 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$ and $\left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right\rangle$ satisfies both the orthogonality and unit norm conditions simultaneously. □

13. The distance between the planes $2x + 3y - 4z = 5$ and $4x + 6y - 8z = 2$ is

   (a) 1

   (b) 2

   (c) $\sqrt{5/2}$

   (d) π

   (e) ∞

   (f) $3/\sqrt{26}$

   (g) 13

   (h) $4/\sqrt{29} \Leftarrow$

   Solution: The planes are parallel with joint normal vector $\mathbf{n} = \langle 2, 3, 4\rangle = \frac{1}{2}\langle 4, 6, 8\rangle$, so it suffices to pick any point on one of the planes and calculate its distance to the second. For example, $(0, 3, 1)$ is in the plane $2x + 3y - 4z = 5$, and its distance from the second is given by the formula

$$\frac{|4(0) + 6(3) - 8(1) - 2|}{\sqrt{4^2 + 6^2 + (-8)^2}} = \frac{4}{\sqrt{29}}.$$

□

14. The equation of a plane parallel to the plane $-3y + 4z = 5$ and 3 units away from it is

   (a) $2x + 5y + 3z = -7$

   (b) $5x - 6y + z = 0$

   (c) $-3y + 8z = 10$

   (d) $-3y + 5z = 16$

   (e) $-3y + 4z = 20 \Leftarrow$

   (f) $2x - 4y + 3z = 2$

   (g) $x = y$

   (h) $z = 0$

   Solution: A parallel plane has the same normal vector $\langle 0, -3, 4\rangle$; the only plane in the solution list with that normal vector is $-3y + 4z = 20$. Check the distance: $(0, 0, 5)$ is a point in the candidate solution plane, and its distance from the plane $-3y + 4z = 5$ is

$$\frac{|-3(0) + 4(5) - 5|}{\sqrt{(-3)^2 + 4^2}} = \frac{15}{5} = 3,$$

as required. □

15. The lines L_1 and L_2 with parametric equations

$$x = 2t, \qquad y = 1 - 4t, \qquad z = -3 + 6t$$

$$x = -s, \qquad y = 2 + 3s, \qquad z = 5 - s$$

are

(a) Parallel lines.

(b) Skew lines. $\Leftarrow$

(c) Lines intersecting at 3 points.

(d) Lines at an angle π to each other.

(e) Lie on a sphere.

(f) None of the above.

Solution: Direction vectors are: $\langle 2, -4, 6\rangle$ for L_1, $\langle -1, 3, -1\rangle$ for L_2. These are not parallel. Check for intersection:

$$\begin{array}{rcl} 2t &=& -s \\ 1-4t &=& 2+3s \\ -3+6t &=& 5-s \end{array} \Rightarrow \begin{array}{rcl} s &=& -1 \\ t &=& 1/2 \\ s &=& 5 \end{array}$$

Since these equations have no simultaneous solution, there is no point of intersection, so lines L_1 and L_2 are skew. □

16. The distance of the point $(-5, 7, 3)$ from the y-axis is

(a) 1

(b) -5

(c) 7

(d) 3

(e) $\sqrt{34/83}$

(f) $\sqrt{34}$ $\Leftarrow$

(g) $\sqrt{84}$

(h) None of the above

Solution: This is just the distance from $(-5, 0, 3)$ from the origin, namely $\sqrt{(-5)^2 + 3^2} = \sqrt{34}$. □

17. Rectangular coordinates $(-2, 2, 0)$ correspond to which spherical coordinates?

(a) $(0, \pi/3, \pi/2)$
(b) $(\sqrt{2}, \pi/4, \pi/2)$
(c) $(\sqrt{2}, -\pi/4, \pi/2)$
(d) $(2, \pi/4, \pi/2)$
(e) $(2, 3\pi/4, \pi/2)$
(f) $(\sqrt{8}, \pi/4, \pi/2)$
(g) $(\sqrt{8}, 3\pi/4, \pi/2) \Leftarrow$
(h) None of the above.

Solution: For $(x, y, z) = (-2, 2, 0)$, compute

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2 + z^2} \\ &= \sqrt{(-2)^2 + 2^2 + 0^2} = \sqrt{8}, \\ \theta &= \tan^{-1} \frac{y}{x} = \tan^{-1}(-1) = 3\pi/4, \\ \phi &= \cos^{-1} \frac{z}{\rho} = \cos^{-1} 0 = \pi/2. \end{aligned}$$

Thus $(\rho, \theta, \phi) = (\sqrt{8}, 3\pi/4, \pi/2)$. □

18. The surface given by the equation $\rho \sin \phi = 5$ in spherical coordinates represents a

(a) Sphere
(b) Cylinder $\Leftarrow$
(c) Cone
(d) Elliptic paraboloid
(e) Hyperboloid of one sheet
(f) Hyperboloid of two sheets
(g) Ellipsoid
(h) Plane

Solution: Rectangular coordinates (x, y, z) satisfy $x = \rho \sin \phi \cos \theta$ and $y = \rho \sin \phi \sin \theta$, so $\rho \sin \phi = \sqrt{x^2 + y^2}$. Hence the equation $\rho \sin \phi = 5$ corresponds to the cylinder $\sqrt{x^2 + y^2} = 5$. □