Ma 233: Calculus III
Solutions to Midterm Examination 2

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18 questions on 18 pages

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1. Find parametric equations for the tangent line at the point $(1/2, -\sqrt{3}/2, -\pi/3)$ on the curve $\mathbf{r}(t) = (\cos t, \sin t, t)$

   (a) $(1/2, -\sqrt{3}/2, -\pi/3) + t\left\langle 1/2, \sqrt{3}/2, 1\right\rangle$

   (b) $(1/2, -\sqrt{3}/2, -\pi/3) + t\left\langle \sqrt{3}/2, 1/2, 1\right\rangle$ $\Leftarrow$

   (c) $(1/2, -\sqrt{3}/2, -\pi/3) + t\left\langle -1/2, \sqrt{3}/2, 1\right\rangle$

   (d) $(1/2, -\sqrt{3}/2, -\pi/3) + t\left\langle -\sqrt{3}/2, 1/2, 1\right\rangle$

   (e) $(1/2, \sqrt{3}/2, 1) + t\left\langle 1/2, -\sqrt{3}/2, -\pi/3\right\rangle$

   (f) $(\sqrt{3}/2, 1/2, 1) + t\left\langle 1/2, -\sqrt{3}/2, -\pi/3\right\rangle$

   (g) $(-1/2, \sqrt{3}/2, 1) + t\left\langle 1/2, -\sqrt{3}/2, -\pi/3\right\rangle$

   (h) $(-\sqrt{3}/2, 1/2, 1) + t\left\langle 1/2, -\sqrt{3}/2, -\pi/3\right\rangle$

Solution: Differentiate to get $\mathbf{r}'(t) = (-\sin t, \cos t, 1)$. The tangent line at $t = -\pi/3$ has direction vector $\mathbf{r}'(-\pi/3) = (-\sin(-\pi/3), \cos(-\pi/3), 1) = (\sqrt{3}/2, 1/2, 1)$ and base point $\mathbf{r}(-\pi/3) = (1/2, -\sqrt{3}/2, -\pi/3)$, so it may be written as

$$(1/2, -\sqrt{3}/2, -\pi/3) + t\left\langle \sqrt{3}/2, 1/2, 1\right\rangle$$

The parameteric equations of the tangent line are thus

$$\begin{aligned} x(t) &= \frac{1}{2} + \frac{\sqrt{3}}{2}t \\ y(t) &= -\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}t \\ z(t) &= -\frac{\pi}{3} + t \end{aligned}$$

□

2. If $\mathbf{r}(t) = \cos(7t)\mathbf{i} + \sin(-5t)\mathbf{j} + 6t\mathbf{k}$, compute $\int_0^{\pi/2} \mathbf{r}(t)\,dt$.

(a) $\frac{1}{7}\mathbf{i} - \frac{1}{5}\mathbf{j} + \frac{3\pi^2}{4}\mathbf{k}$.

(b) $\frac{1}{7}\mathbf{i} + \frac{1}{5}\mathbf{j} + \frac{3\pi^2}{4}\mathbf{k}$.

(c) $\frac{1}{7}\mathbf{i} - \frac{1}{5}\mathbf{j} - \frac{3\pi^2}{4}\mathbf{k}$.

(d) $-\frac{1}{7}\mathbf{i} - \frac{1}{5}\mathbf{j} + \frac{3\pi^2}{4}\mathbf{k}$. $\Leftarrow$

(e) $\mathbf{i} - \mathbf{j} + 3\pi\mathbf{k}$.

(f) $\mathbf{i} + \mathbf{j} + 3\pi\mathbf{k}$.

(g) $\mathbf{i} - \mathbf{j} - 3\pi\mathbf{k}$.

(h) $-\mathbf{i} - \mathbf{j} + 3\pi\mathbf{k}$.

Solution: Integrate the components separately to get the antiderivative

$$\frac{1}{7}\sin(7t)\mathbf{i} + \frac{1}{5}\cos(-5t)\mathbf{j} + \frac{6t^2}{2}\mathbf{k}$$

Evaluating the difference between 0 and $\pi/2$ gives

$$-\frac{1}{7}\mathbf{i} - \frac{1}{5}\mathbf{j} + \frac{3\pi^2}{4}\mathbf{k}$$

□

3. Find the length of the curve

$$\{\mathbf{r}(t) = \cos(5t)\mathbf{i} + \sin(5t)\mathbf{j} + 6t\mathbf{k} : -2 \le t \le 6\}$$

(a) 8

(b) 48

(c) $2\sqrt{61}$

(d) $4\sqrt{61}$

(e) $8\sqrt{61}$ $\Leftarrow$

(f) $2\sqrt{86}$

(g) $4\sqrt{86}$

(h) $8\sqrt{86}$

Solution: Find $\mathbf{r}'(t) = -5\sin(5t)\mathbf{i} + 5\cos(5t)\mathbf{j} + 6\mathbf{k}$ and

$$\|\mathbf{r}'(t)\| = \sqrt{25\sin^2(5t) + 25\cos^2(5t) + 36} = \sqrt{61}.$$

Integrate this to get the length

$$\int_{-2}^{6} \|\mathbf{r}'(t)\|\, dt = 8\sqrt{61}.$$

□

4. Given that a point has acceleration $\mathbf{a}(t) = \langle 1, 2t, -3t+1\rangle$, its position is $\langle 1,1,1\rangle$ at $t=0$ and its velocity is $\langle -2,-2,-2\rangle$ at $t=0$, find its position at all times t.

(a) $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 - t + 2, \quad \frac{1}{3}t^3 - t + 2, \quad -\frac{1}{2}t^3 + \frac{1}{2}t^2 - t + 2\right\rangle$

(b) $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 + t + 2, \quad \frac{1}{3}t^3 + t + 2, \quad -\frac{1}{2}t^3 + \frac{1}{2}t^2 + t + 2\right\rangle$

(c) $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 + t - 2, \quad \frac{1}{3}t^3 + t - 2, \quad -\frac{1}{2}t^3 + \frac{1}{2}t^2 + t - 2\right\rangle$

(d) $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 - t - 2, \quad \frac{1}{3}t^3 - t - 2, \quad -\frac{1}{2}t^3 + \frac{1}{2}t^2 - t - 2\right\rangle$

(e) $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 - 2t + 1, \quad \frac{1}{3}t^3 - 2t + 1, \quad -\frac{1}{2}t^3 + \frac{1}{2}t^2 - 2t + 1\right\rangle$ $\Leftarrow$

(f) $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 + 2t + 1, \quad \frac{1}{3}t^3 + 2t + 1, \quad -\frac{1}{2}t^3 + \frac{1}{2}t^2 + 2t + 1\right\rangle$

(g) $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 + 2t - 1, \quad \frac{1}{3}t^3 + 2t - 1, \quad -\frac{1}{2}t^3 + \frac{1}{2}t^2 + 2t - 1\right\rangle$

(h) $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 - 2t - 1, \quad \frac{1}{3}t^3 - 2t - 1, \quad -\frac{1}{2}t^3 + \frac{1}{2}t^2 - 2t - 1\right\rangle$

Solution: Velocity, the antiderivative of acceleration, is

$$\mathbf{v}(t) = \left\langle t + c_x, \quad t^2 + c_y, \quad -\frac{3}{2}t^2 + t + c_z\right\rangle,$$

where c_x, c_y, c_z are constants of integration. Determine these from the condition $\langle -2,-2,-2\rangle = \mathbf{v}(0) = \langle c_x, c_y, c_z\rangle$, so $c_x = -2$, $c_y = -2$, $c_z = -2$ and $\mathbf{v}(t) = \left\langle t-2, \quad t^2-2, \quad -\frac{3}{2}t^2 + t - 2\right\rangle$.

Position, the antiderivative of velocity, is

$$\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 - 2t + k_x, \quad \frac{1}{3}t^3 - 2t + k_y, \quad -\frac{1}{2}t^3 + \frac{1}{2}t^2 - 2t + k_z\right\rangle,$$

where k_x, k_y, k_z are constants of integration. Determine these from the condition $\langle 1,1,1\rangle = \mathbf{r}(0) = \langle k_x, k_y, k_z\rangle$, so $k_x = 1$, $k_y = 1$, $k_z = 1$ and

$$\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 - 2t + 1, \quad \frac{1}{3}t^3 - 2t + 1, \quad -\frac{1}{2}t^3 + \frac{1}{2}t^2 - 2t + 1\right\rangle$$

□

5. Find the curvature of $y = \cos(3x)$ at $x = \pi/4$

(a) $3/2$

(b) $2/3$

(c) $11/13^{2/3}$

(d) $11/13^{3/2}$

(e) $13/11^{2/3}$

(f) $13/18^{3/2}$

(g) $18/11^{2/3}$

(h) $18/11^{3/2}$ $\Leftarrow$

(i) $11/18^{3/2}$

(j) $13/11^{3/2}$

Solution: This is the special case of a curve being the graph of a function: $y = f(x) = \cos(3x)$. Therefore, use the formula

$$\kappa(x) = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}$$

But $f'(x) = -3\sin(3x)$ and $f''(x) = -9\cos(3x)$, so the formula expands to

$$\kappa(x) = \frac{|9\cos(3x)|}{(1 + 9\sin^2(3x))^{3/2}}$$

Thus

$$\kappa(\pi/4) = \frac{|-9/\sqrt{2}|}{(1 + 9/2)^{3/2}} = \frac{18}{11^{3/2}}$$

□

6. Let $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 2t\mathbf{k}$. Find the unit binormal vector $\mathbf{B}(t)$ for all t.

(a) $(-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + 2\mathbf{k}$

(b) $(\sin t)\mathbf{i} + (-\cos t)\mathbf{j} + 2\mathbf{k}$

(c) $\frac{-\sin t}{\sqrt{5}}\mathbf{i} + \frac{\cos t}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$

(d) $\frac{\sin t}{\sqrt{5}}\mathbf{i} + \frac{\cos t}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$

(e) $\frac{-\cos t}{\sqrt{5}}\mathbf{i} + \frac{-\sin t}{\sqrt{5}}\mathbf{j} + 0\mathbf{k}$

(f) $\frac{-\cos t}{\sqrt{5}}\mathbf{i} + \frac{\sin t}{\sqrt{5}}\mathbf{j} + 0\mathbf{k}$

(g) $-\cos(t)\mathbf{i} - \sin(t)\mathbf{j} + 0\mathbf{k}$

(h) $\cos(t)\mathbf{i} + \sin(t)\mathbf{j} + 0\mathbf{k}$

(i) $\frac{-2\sin t}{\sqrt{5}}\mathbf{i} + \frac{-2\cos t}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$

(j) $\frac{2\sin t}{\sqrt{5}}\mathbf{i} + \frac{-2\cos t}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$ $\Leftarrow$

Solution: Tangent vector: $\mathbf{r}'(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + 2\mathbf{k}$. Its length is the constant $\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{5}$ for all t, so the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-\sin t}{\sqrt{5}}\mathbf{i} + \frac{\cos t}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}.$$

Normal vector:

$$\mathbf{T}'(t) = \frac{-\cos t}{\sqrt{5}}\mathbf{i} + \frac{-\sin t}{\sqrt{5}}\mathbf{j} + 0\mathbf{k},$$

and its length is

$$\|\mathbf{T}'(t)\| = \sqrt{(-\cos t)^2/5 + (\sin t)^2/5} = 1/\sqrt{5}.$$

Hence the unit normal vector is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos(t)\mathbf{i} - \sin(t)\mathbf{j} + 0\mathbf{k}.$$

Binormal vector:

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{2\sin t}{\sqrt{5}}\mathbf{i} + \frac{-2\cos t}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$$

□

7. Match the following surfaces with the verbal description of their level curves:

1. $z = \sqrt{9 - x^2 - y^2}$	X. a collection of parallel lines
2. $z = y^2 - x^2$	Y. a collection of circles
3. $z = \dfrac{1}{y} - 3$	Z. two lines and a collection of hyperbolas

(a) 1 is X; 2 is Y; 3 is Z.

(b) 1 is Y; 2 is Z; 3 is X. ⇐

(c) 1 is Z; 2 is X; 3 is Y.

(d) 1 is X; 2 is Z; 3 is Y.

(e) 1 is Z; 2 is Y; 3 is X.

(f) 1 is Y; 2 is X; 3 is Z.

(g) All of them are X.

(h) All of them are Y.

(i) All of them are Z.

(j) None of the above.

Solution: Equation 1 is a hemisphere, with circles (Y) as level sets. Equation 2 is a hyperbolic paraboloid, with lines and hyperbolas (Z) as level sets. Equation 3 is hyperbolic cylinder with parallel lines (X) as level sets. □

8. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - 3xy + y^2}{(x-y)^2}$, if it exists.

(a) -4

(b) -3

(c) -2

(d) -1

(e) 0

(f) 1

(g) 2

(h) 3

(i) 4

(j) The limit does not exist. $\Leftarrow$

Solution: The limit does not exist. Along the line $y = 0$ through $(0,0)$, the ratio is constantly 1, while along the line $y = -x$ through $(0,0)$, the ratio is constantly $\frac{5}{4}$. Since these do not agree at $(0,0)$, there can be no limit at $(0,0)$. $\square$

9. Find the partial derivative f_{xxy} for the function $f(x, y) = e^{xy^2}$.

(a) $y^6 e^{xy^2}$

(b) $4y^3 e^{xy^2}$

(c) $4y^3 e^{xy^2} + 2y^5 e^{xy^2}$

(d) $4y^3 e^{xy^2} + 2xy^5 e^{xy^2}$ $\Leftarrow$

(e) $2ye^{xy^2} + 3xy^5 e^{xy^2}$

(f) $2ye^{xy^2} + 2xy^3 e^{xy^2}$

(g) $3xy^3 e^{xy^2}$

(h) $y^5 x e^{xy^2}$

(i) The derivative exists, but is none of the above.

(j) The derivative does not exist.

Solution: $f_x = y^2 e^{xy^2}$; $f_{xx} = y^4 e^{xy^2}$; $f_{xxy} = 4y^3 e^{xy^2} + 2xy^5 e^{xy^2}$; $\square$

10. Determine whether each of the following functions is a solution to Laplace's equation $u_{xx} + u_{yy} = 0$.

(I) $u(x, y) = x^2 + y^2$

(II) $u(x, y) = x^2 - y^2$

(III) $u(x, y) = \log \sqrt{x^2 + y^2}$

(a) I only.

(b) II only.

(c) III only.

(d) IV only.

(e) I and II only.

(f) I and III only.

(g) II and III only. $\Leftarrow$

(h) All.

(i) None.

Solution: For (I): $u_x = 2x$, $u_{xx} = 2$; $u_y = 2y$, $u_{yy} = 2$. Thus $u_{xx} + u_{yy} = 4 \neq 0$.

For (II): $u_x = 2x$, $u_{xx} = 2$; $u_y = -2y$, $u_{yy} = -2$. Thus $u_{xx} + u_{yy} = 0$.

For (III): $u_x = x/(x^2 + y^2)$, $u_{xx} = (y^2 - x^2)/(x^2 + y^2)^2$; $u_y = y/(x^2 + y^2)$, $u_{yy} = (x^2 - y^2)/(x^2 + y^2)^2$; Thus $u_{xx} + u_{yy} = 0$. □

11. Find the equation of the tangent plane to the surface $z = x^2 - y^2$ at the point $(2, 1, 3)$.

(a) $2x - y - z = 0$

(b) $2x - y - z = 3$

(c) $4x - 2y - z = 0$

(d) $4x - 2y - z = 3$ $\Leftarrow$

(e) $4x - 2y - z = 6$

(f) $8x - 4y - 2z = 3$

(g) $x - 2y - \frac{1}{2}z = 3$

(h) $x - 2y + \frac{1}{2}z = 3$

(i) $x - 2y - \frac{1}{2}z = 6$

(j) $x - 2y + \frac{1}{2}z = 6$

Solution: This surface is a level surface for the function $F(x, y, z) = x^2 - y^2 - z = 0$, so its tangent plane at $(2, 1, 3)$ has $\nabla F(2, 1, 3) = \langle 2x, -2y, -1\rangle \,|_{(2,1,3)} = \langle 4, -2, -1\rangle$ as a normal vector. Since $(2, 1, 3)$ is in the tangent plane, its equation is $4x - 2y - z = \langle 4, -2, -1\rangle \cdot \langle 2, 1, 3\rangle = 3$. $\square$

12. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

(a) 8.0 cm^3

(b) 8.1 cm^3

(c) 8.2 cm^3

(d) 8.3 cm^3

(e) 8.4 cm^3

(f) 8.5 cm^3

(g) 8.6 cm^3

(h) 8.7 cm^3

(i) 8.8 cm^3 $\Leftarrow$

(j) 8.9 cm^3

Solution: Let $h = 10$ cm denote the height of the can, and $d = 4$ cm denote its diameter. The area of the top and bottom together is $2 \times \pi(d/2)^2 = 8\pi \approx 25$, while the area of the side is $h \times \pi d = 40\pi \approx 126$, so the total metal in the can is approximately $(0.10)(25) + (0.05)(126) = 8.8$ $\square$

13. Let $z = f(x - y)$ for a differentiable function f. Then $\partial z/\partial x + \partial z/\partial y$ is

(a) 1

(b) 2

(c) $\sqrt{5/2}$

(d) π

(e) ∞

(f) 0 $\Leftarrow$

(g) -1

(h) -2

Solution: By the chain rule, $\partial z/\partial x = f'(x - y)$ while $\partial z/\partial y = (-1)f'(x - y)$. Thefore, $\partial z/\partial x + \partial z/\partial y = 0$ $\square$

14. Find the directional derivative of $g(x, y, z) = 3e^x \cos(yz)$ at the point $P(0, 0, 0)$ in the direction $\left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right\rangle$.

(a) 2 $\Leftarrow$

(b) 1

(c) -3

(d) 5

(e) 0

(f) -4

(g) ∞

(h) 3.43

Solution: Compute the gradient at $P(0, 0, 0)$:

$$\nabla g(0,0,0) = \langle 3e^x \cos(yz), -3ze^x \sin(yz), -3ye^x \sin(yz) \rangle \, |_{(0,0,0)} = \langle 3, 0, 0 \rangle \, .$$

Evaluate the dot product to get the directional derivative in the direction $\mathbf{u} = \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right\rangle$:

$$D_{\mathbf{u}} \, g(0,0,0) = \nabla g(0,0,0) \cdot \mathbf{u} = \langle 3, 0, 0 \rangle \cdot \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right\rangle = 2.$$

□

15. Consider the surface $7x^2 - 3y^2 + z^2 = 8$. Find an equation for the plane tangent to this surface at its point $P(1, 1, 2)$.

(a) $7x - 3y + 2z = 8$ $\Leftarrow$

(b) $7x - 3y + z = 8$

(c) $-3y + 8z = 10$

(d) $-3y + 5z = 16$

(e) $-3y + 4z = 20$

(f) $2x - z = 2$

(g) $x = y$

(h) $z = 0$

(i) None of the above

Solution: The surface is a level set for the function $f(x, y, z) = 7x^2 - 3y^2 + z^2$. This function is a polynomial, hence it is differentiable and its gradient $\nabla f(1, 1, 2)$ is a normal vector for the tangent plane at $P(1, 1, 2)$. But

$$\nabla f(1, 1, 2) = \langle 14x, -6y, 2z \rangle |_{(1,1,2)} = \langle 14, -6, 4 \rangle ,$$

so the equation of the tangent plane is

$$\langle 14, -6, 4 \rangle \cdot \langle x, y, z \rangle = \langle 14, -6, 4 \rangle \cdot \langle 1, 1, 2 \rangle$$

which simplifies to the equation $7x - 3y + 2z = 8$.

$\square$

16. The volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$ is

(a) $3/4$

(b) 5

(c) $7/2$

(d) $4/3$ $\Leftarrow$

(e) $\sqrt{34/83}$

(f) $\sqrt{34}$

(g) $\sqrt{84}$

(h) None of the above

Solution: Let x, y, z be the coordinates of the vertex on the plane $x + 2y + 3z = 6$. The dimensions of the box will then be x, y, z and its volume will be

$$xyz = (6 - 2y - 3z)yz \stackrel{\text{def}}{=} f(y, z) = 6yz - 2y^2z - 3yz^2$$

To maximize this function over all y, z that occur in the first octant, first find the critical points:

$$\begin{aligned} \nabla f(y,z) &= \left\langle 6z - 4yz - 3z^2,\ 6y - 2y^2 - 6yz \right\rangle = \langle 0, 0 \rangle \\ \iff &\quad 6z - 4yz - 3z^2 = 0 \text{ and } 6y - 2y^2 - 6yz = 0 \\ \iff &\quad z(6 - 4y - 3z) = 0 \text{ and } y(6 - 2y - 6z) = 0. \end{aligned}$$

If either $z = 0$ or $y = 0$, then the volume of box will be 0, clearly the absolute minimum. This accounts for all points x, y, z on one of the coordinate planes, too; they are all absolute minima for the volume.

If both z and y are nonzero, then $\nabla f(y, z) = \langle 0, 0 \rangle$ only if $6 - 4y - 3z = 0$ and $6 - 2y - 6z = 0$. But this is a linear system of two equations in two unknowns:

$$\begin{aligned} 4y + 3z &= 6 \\ 2y + 6z &= 6 \end{aligned}$$

Subtracting twice the second equation from the first and solving the result yields $z = 2/3$, so $y = 1$. This must be the unique absolute maximum, as it is a critical point and the positive value of $g(y, z)$ there exceeds the zero value at all the other critical points.

The corresponding third dimension of the box will be $x = 6 - 2y - 3z = 2$, so the box will have volume $(2)(1)(\frac{2}{3}) = \frac{4}{3}$. $\square$

17. The points on the hyperboloid $x^2 - y^2 + 2z^2 = 1$ where the normal line is parallel to the line that joins the points $(3, -1, 0)$ and $(5, 3, 6)$ are

(a) $(3, -1.0)$ and $(5, 3, 6)$

(b) $(\sqrt{35}, \sqrt{38}, \sqrt{2})$ and $(-\sqrt{35}, -\sqrt{38}, \sqrt{2})$

(c) $(2, 2, 2)$ and $(1, 1, 1)$

(d) $(\sqrt{6}, 3, -\sqrt{2})$ and $(\sqrt{6}, 3, \sqrt{2})$

(e) any two points.

(f) $(1, 0, 0)$ and $(0, 0, 1/\sqrt{2})$

(g) $(\sqrt{6}/3, -2\sqrt{6}/3, \sqrt{6}/2)$ and $(-\sqrt{6}/3, 2\sqrt{6}/3, -\sqrt{6}/2)$ $\Leftarrow$

(h) None of the above.

Solution: The normal line is given by the gradient of $f(x, y, z) = x^2 - y^2 + 2z^2$ for which the hyperboloid is a level set. But

$$\nabla f(x, y, z) = \langle 2x, -2y, 4z \rangle$$

The line joining the points $(3, -1, 0)$ and $(5, 3, 6)$ has direction vector $\mathbf{u} = (3, -1, 0) - (5, 3, 6) = \langle 2, 4, 6 \rangle$. The normal is parallel to this line if and only if $\nabla f(x, y, z) \times \mathbf{u} = \mathbf{0}$, or

$$\langle 2x, -2y, 4z \rangle \times \langle 2, 4, 6 \rangle = \langle 0, 0, 0 \rangle \iff \begin{cases} 8x + 4y &= 0 \\ -12y - 16z &= 0 \\ -12x + 8z &= 0 \end{cases}$$
$$\iff \begin{cases} 2x + y &= 0 \\ 3y + 4z &= 0 \\ 3x - 2z &= 0 \end{cases}$$

Substituting $y = -2x$ from the first equation and $z = \frac{3}{2}x$ from the third equation into the second equation yields $-6x + 6x = 0$, so the system is consistent and any points on the parameterized line $\{x = t, y = -2t, z = \frac{3}{2}t\}$ will solve it. Substitute these parametric formulas into the equation of the hyperboloid to find the desired point:

$$1 = f(t, -2t, \frac{3}{2}t) = (t)^2 - (-2t)^2 + 2(\frac{3}{2}t)^2 = \frac{3}{2}t^2,$$

so $t = \pm\sqrt{\frac{2}{3}}$, giving the points $(x, y, z) = (\sqrt{\frac{2}{3}}, -2\sqrt{\frac{2}{3}}, \frac{3}{2}\sqrt{\frac{2}{3}}) = (\sqrt{6}/3, -2\sqrt{6}/3, \sqrt{6}/2)$ and $(x, y, z) = (-\sqrt{6}/3, 2\sqrt{6}/3, -\sqrt{6}/2)$. □

18. A flat circular plate has the shape of the region $\{(x,y) : x^2 + y^2 \leq 1\}$. The plate is cooled so that the temperature at the point (x,y) is

$$T(x,y) = x^2 + 2y^2 - x$$

The extreme temperatures on the surface are

(a) 3.17 and 2.16

(b) 2.25 and -0.25 $\Leftarrow$

(c) 4.33 and -4.33

(d) 5.15 and 0

(e) 0.91 and 0.19

(f) 1.67 and -3.20

(g) 3.33 and -4.04

(h) 0.1 and 0.1

Solution: Look for critical points: $\mathbf{0} = \nabla T(x,y) = \langle 2x-1, 4y \rangle$ implies $x = \frac{1}{2}$, $y = 0$. There, the temperature is $T(\frac{1}{2}, 0) = -0.25$.

Look on the boundary. Substituting the circle parameterization $x = \cos t$, $y = \sin t$ for the boundary points gives $f(t) = T(\cos t, \sin t) = \cos^2 t + 2\sin^2 t - \cos t$. This function is extremal at its critical points $0 = f'(t) = -2\cos t \sin t + 4 \sin t \cos t + \sin t$, where

$$\sin t(2\cos t + 1) = 0 \qquad \Longleftrightarrow \qquad y(2x+1) = 0.$$

This is true on the circle $x^2 + y^2 = 1$ if and only if either $y = 0$ and $x = \pm 1$, or $x = -\frac{1}{2}$ so $y = \pm\frac{\sqrt{3}}{2}$. The corresponding temperatures are:

$$T(1,0) = 0; \quad T(-1,0) = 2; \quad T(-\frac{1}{2}, \frac{\sqrt{3}}{2})) = 2.25; \qquad T(-\frac{1}{2}, -\frac{\sqrt{3}}{2})) = 2.25$$

The extremal values from these five candidates are $T = -0.25$ and $T = 2.25$. $\square$