Ma 233: Calculus III
Solutions to Midterm Examination 3

Profs. Krishtal, Ravindra, and Wickerhauser
17 questions on 17 pages

Tuesday, April 12th, 2005

1. Evaluate the iterated integral
$$\int_1^2 \int_1^2 (x+2y)^{-3}\,dxdy$$

(a) $\frac{1}{4}$
(b) $\frac{1}{15}$
(c) $\frac{1}{20}$
(d) $\frac{3}{20}$
(e) $\frac{1}{24}$
(f) $\frac{1}{60}$
(g) $\frac{1}{80}$ $\Leftarrow$
(h) $\frac{1}{240}$

Solution:

$$\begin{aligned}
\int_{y=1}^2 \int_{x=1}^2 (x+2y)^{-3}\,dxdy &= \int_{y=1}^2 \left[\frac{(x+2y)^{-2}}{-2}\right]_{x=1}^2 dy \\
&= \int_{y=1}^2 \left[\frac{(2+2y)^{-2}}{-2} - \frac{(1+2y)^{-2}}{-2}\right] dy \\
&= \left[\frac{(2+2y)^{-1}}{(-2)(-1)(2)} - \frac{(1+2y)^{-1}}{(-2)(-1)(2)}\right]_1^2 \\
&= \frac{1}{4}\left[(2+4)^{-1} - (2+2)^{-1} + (1+4)^{-1} - (1+2)^{-1}\right]_1^2 = \frac{1}{80}.
\end{aligned}$$

□

2. Calculate the volume under the elliptic paraboloid $z = 2x^2 + 3y^2$ and over the rectangle $R = [-2, 2] \times [-3, 3]$.

   (a) 180
   (b) 210
   (c) 240
   (d) 270
   (e) 280 $\Leftarrow$
   (f) 300
   (g) 320
   (h) 360

Solution: The volume is the integral of the height function $f(x, y) = z - 0$ over the base rectangle R. This may be evaluated by iterated integration:

$$\begin{aligned} \int_{x=-2}^{2} \int_{y=-3}^{3} (2x^2 + 3y^2)\, dy dx &= \int_{x=-2}^{2} \Big[2x^2 y + y^3\Big]_{-3}^{3}\, dx \\ &= \int_{x=-2}^{2} \Big[12x^2 + 54\Big]\, dx = \Big[4x^3 + 54x\Big]_{x=-2}^{2} = 280. \end{aligned}$$

□

3. Evaluate the double integral
$$I = \iint_D xy\, dA$$
where D is the triangular region with vertices $(0,0), (2,0), (2,1)$.

(a) 1

(b) 1/2 $\Leftarrow$

(c) 1/3

(d) 1/4

(e) 1/5

(f) 1/6

(g) 1/8

(h) 1/16

Solution: Write $D = \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq x/2\}$ and compute the double integral by Fubini's theorem as an iterated integral in the y-variable first:

$$I = \int_{x=0}^{2} \int_{y=0}^{x/2} xy\, dydx = \int_{x=0}^{2} \left[\frac{xy^2}{2}\right]_0^{x/2} dx = \int_{x=0}^{2} \left[\frac{x^3}{8}\right] dx = \left[\frac{x^4}{32}\right]_0^2 = \frac{1}{2}.$$

Also by Fubini's theorem, the iterated integral in the other order gives the same result. $\square$

4. Evaluate the integral by reversing the order of integration.

$$\int_{y=0}^{2}\int_{x=y}^{2} e^{-x^2}\,dxdy$$

(a) $(1-e^{-4})/2$ $\Leftarrow$

(b) $(1-e^{-2})/2$

(c) $(1-e^{-2})/4$

(d) $(e^{-2}-e^{-4})/2$

(e) $(e^{-4}-e^{-2})/2$

(f) $(e^{-4}-e^{-2})/4$

(g) $(e^{-4}-e^{-2})/3$

(h) $1-e^{-4}$

Solution: The domain of integration has the two descriptions $\{0 \le y \le 2, y \le x \le 2\}$ and $\{0 \le x \le 2, 0 \le y \le x\}$. Use the second to compute the iterated integral in the other order as suggested:

$$\int_{y=0}^{2}\int_{x=y}^{2} e^{-x^2}\,dxdy = \int_{x=0}^{2}\int_{y=0}^{x} e^{-x^2}\,dydx = \int_{x=0}^{2} xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}\Big|_0^2 = (1-e^{-4})/2.$$

□

5. Using polar coordinates, evaluate the integral

$$\iint_R \cos(x^2 + y^2)\, dA$$

where R is the region $4 \le x^2 + y^2 \le 9$.

(a) $2\pi \sin 9$

(b) $2\pi \sin 5$

(c) $2\pi \sin 4$

(d) $2\pi \sin 1$

(e) $\pi \sin 9$

(f) $2\pi(\sin 9 - \sin 4)$

(g) $\pi(\sin 9 - \sin 4)$ $\Leftarrow$

(h) $2\pi(\sin 3 - \sin 2)$

(i) $\pi(\sin 3 - \sin 2)$

Solution: Let $x = r\cos\theta$, $y = r\sin\theta$, so $x^2 + y^2 = r^2$. In these polar coordinates (r, θ), $dA = r\, drd\theta$ and $R = \{(r, \theta) : 2 \le r \le 3\}$, giving

$$\iint_R \cos(x^2 + y^2) dA = \int_{\theta=0}^{2\pi} \int_{r=2}^{3} \cos(r^2) r\, drd\theta = 2\pi \left[\frac{\sin(r^2)}{2}\right]_2^3 = \pi[\sin 9 - \sin 4].$$

□

6. Let X, Y be two random variables and $f(x,y) = Ce^{-(x^2+y^2)}$ be the joint probability density function. Find the constant C and the probability P of (X,Y) being within the unit circle.

(a) $C = 0, \quad P = 1$

(b) $C = 1, \quad P = \pi(1 - e^{-1})$

(c) $C = 1, \quad P = \pi(e^{-1})$

(d) $C = 1, \quad P = 1/2\pi$

(e) $C = 1/\sqrt{\pi}, \quad P = (1 - e^{-1})\sqrt{\pi}$

(f) $C = 1/\sqrt{\pi}, \quad P = e^{-1}\sqrt{\pi}$

(g) $C = 1/\sqrt{\pi}, \quad P = 1/2\sqrt{\pi}$

(h) $C = 1/\pi, \quad P = 1 - e^{-1}$ $\Leftarrow$

(i) $C = 1/\pi, \quad P = e^{-1}$

(j) $C = 1/\pi, \quad P = 1/2$

Solution: Since the joint probability density function is $f(x,y) = Ce^{-(x^2+y^2)}$, we must have

$$1 = \iint_{R^2} f(x,y)\, dA = \iint_{R^2} Ce^{-(x^2+y^2)}\, dA = C\int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr d\theta = C\pi.$$

Hence $C = 1/\pi$ and

$$P(X^2 + Y^2 \leq 1) = \iint_{x^2+y^2\leq 1} f(x,y)\, dA = \frac{1}{\pi}\int_0^{2\pi}\int_0^{1} e^{-r^2} r\, dr d\theta = 1 - e^{-1}.$$

□

7. Find the *surface area* of the bounded region below the sphere $x^2 + y^2 + (z - 1)^2 = 1$ and above the cone $z^2 = x^2 + y^2$.

(a) $(1 + 1/\sqrt{2})\pi$

(b) $(2 + 1/\sqrt{2})\pi$

(c) $(3 + 1/\sqrt{2})\pi$

(d) $(1 + \sqrt{2})\pi$

(e) $(2 + \sqrt{2})\pi$ $\Leftarrow$

(f) $(3 + \sqrt{2})\pi$

(g) π

(h) 2π

(i) 3π

Solution: The boundary surfaces, the sphere and the cone, intersect in a circle $\{x^2 + y^2 = 1, z = 1\}$. The center of the sphere is $(0, 0, 1)$, so the upper part of the surface is exactly a half-sphere of radius 1. Thus its area is $4\pi(1^2)/2 = 2\pi$.

To find the surface area of the lower part, the cone, use spherical coordinates. The equation of the cone becomes $\phi = \pi/4$ and, therefore, the appropriate parametrization is

$$\mathbf{r}(\rho, \theta) = \rho \sin\phi \cos\theta \, \mathbf{i} + \rho \sin\phi \sin\theta \, \mathbf{j} + \rho \cos\phi \, \mathbf{k} = \frac{\rho}{\sqrt{2}} \left[\cos\theta \, \mathbf{i} + \sin\theta \, \mathbf{j} + \mathbf{k}\right],$$

for $0 \le \theta \le 2\pi$, $0 \le \rho \le \sqrt{2}$. Now we have

$$\begin{aligned} \mathbf{r}_\rho(\rho, \theta) &= \frac{1}{\sqrt{2}} \left[\cos\theta \, \mathbf{i} + \sin\theta \, \mathbf{j} + 1 \, \mathbf{k}\right], \\ \mathbf{r}_\theta(\rho, \theta) &= \frac{\rho}{\sqrt{2}} \left[-\sin\theta \mathbf{i} + \cos\theta \mathbf{j} + 0 \, \mathbf{k}\right]. \end{aligned}$$

Hence, $\mathbf{r}_\rho \times \mathbf{r}_\theta = \frac{\rho}{2} \left[-\cos\theta \, \mathbf{i} - \sin\theta \, \mathbf{j} + 1 \, \mathbf{k}\right]$, and $|\mathbf{r}_\rho \times \mathbf{r}_\theta| = \rho/\sqrt{2}$.

Now, using the formula for the surface area we get

$$S(cone) = \int_0^{2\pi} \int_0^{\sqrt{2}} \frac{\rho}{\sqrt{2}} \, d\rho d\theta = (2\pi) \frac{\rho^2}{2\sqrt{2}} \Big|_0^{\sqrt{2}} = \pi\sqrt{2}$$

Thus, the total area is $2\pi + \pi\sqrt{2} = (2 + \sqrt{2})\pi$. □

8. Evaluate the triple integral

$$\iiint_E (x^2 + y^2 + z^2)\, dV,$$

where $E = \{x^2 + y^2 \leq 1,\ |z| \leq 1\}$.

(a) 2π

(b) $3\pi/2$

(c) $2\pi/3$

(d) $3\pi/4$

(e) $4\pi/3$

(f) $3\pi/5$

(g) $5\pi/3$ $\Leftarrow$

(h) $3\pi/7$

(i) $7\pi/3$

Solution: This is easier in cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, z, for then

$$E = \{0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi, \quad -1 \leq z \leq -1\}.$$

The integrand becomes $x^2 + y^2 + z^2 = r^2 + z^2$ and the volume element is $dV = r\, dr d\theta dz$ in the (r, θ, z) coordinates, so the integral may be evaluated by iteration:

$$\begin{aligned}
\iiint_E (x^2 + y^2 + z^2)\, dV &= \int_{-1}^{1} \int_0^{2\pi} \int_0^1 (r^2 + z^2) r\, dr d\theta dz \\
&= \int_{-1}^{1} \int_0^{2\pi} \left[\frac{r^4}{4} + \frac{z^2 r^2}{2} \right]_{r=0}^{1} d\theta dz \\
&= \int_{-1}^{1} \int_0^{2\pi} \left[\frac{1}{4} + \frac{z^2}{2} \right] d\theta dz \\
&= \int_{-1}^{1} \pi \left[\frac{1}{2} + z^2 \right] dz \\
&= \pi \left[\frac{z}{2} + \frac{z^3}{3} \right]_{-1}^{1} = \pi \left[1 + \frac{2}{3} \right] = 5\pi/3
\end{aligned}$$

□

9. Evaluate the Jacobian of the transformation

$$x = \arctan \frac{u}{v}, \quad y = \ln(u^2 + v^2)$$

at $u = v = 1$.

(a) 16

(b) 8

(c) 4

(d) 2

(e) 1 $\Leftarrow$

(f) 0.5

(g) 0.25

(h) 0.125

(i) 0.0625

Solution: Compute the partial derivatives at $(u, v) = (1, 1)$:

$$\begin{aligned}
\frac{\partial x}{\partial u}(1,1) &= \left(\frac{1}{v}\right) \frac{1}{1 + (u/v)^2}\bigg|_{(1,1)} &= \frac{1}{2}; \\
\frac{\partial x}{\partial v}(1,1) &= \left(-\frac{u}{v^2}\right) \frac{1}{1 + (u/v)^2}\bigg|_{(1,1)} &= -\frac{1}{2}; \\
\frac{\partial y}{\partial u}(1,1) &= \frac{2u}{u^2 + v^2}\bigg|_{(1,1)} &= 1; \\
\frac{\partial y}{\partial v}(1,1) &= \frac{2v}{u^2 + v^2}\bigg|_{(1,1)} &= 1.
\end{aligned}$$

Hence

$$\frac{\partial(x, y)}{\partial(u, v)} = (\frac{1}{2})(1) - (-\frac{1}{2})(1) = 1.$$

□

10. Find the area of the image in the xy-plane of the annulus

$$R = \{(u,v): \ 1 \leq u^2 + v^2 \leq 4\}$$

under the transformation:

$$x = \arctan \frac{u}{v}, \quad y = \ln(u^2 + v^2).$$

(a) π

(b) 2π

(c) 4π

(d) $\pi \ln 2$

(e) $2\pi \ln 2$

(f) $4\pi \ln 2$ $\Leftarrow$

(g) $\ln 2$

(h) $2 \ln 2$

(i) $4 \ln 2$

Solution: Compute the partial derivatives:

$$\begin{aligned} \frac{\partial x}{\partial u} &= \left(\frac{1}{v}\right) \frac{1}{1+(u/v)^2}; & \frac{\partial x}{\partial v} &= \left(-\frac{u}{v^2}\right) \frac{1}{1+(u/v)^2}; \\ \frac{\partial y}{\partial u} &= \frac{2u}{u^2+v^2}; & \frac{\partial y}{\partial v} &= \frac{2v}{u^2+v^2}. \end{aligned}$$

Hence the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{2}{u^2+v^2},$$

The area of the region in the xy-plane that is the image of R under the transformation $(u,v) \mapsto (x,y)$ is given by

$$\iint_R \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, dudv$$

Since R is an annulus, it has a simple description in polar coordinates: $R = \{1 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi\}$. But then the integral may be solved by iteration:

$$\iint_R \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, dudv = \int_0^{2\pi} \int_1^2 \frac{2}{r^2} r \, dr d\theta = 4\pi \ln r \Big|_1^2 = 4\pi \ln 2.$$

$\square$

11. Estimate the value of the line integral $\int_C x\,ds$, where C is the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.

(a) $\dfrac{5\sqrt{5}-1}{12} \approx .848$ $\Leftarrow$

(b) $\dfrac{12}{5\sqrt{5}+1} \approx .985$

(c) $\dfrac{6}{5\sqrt{5}+1} \approx .492$

(d) $\dfrac{6}{5\sqrt{5}} \approx .537$

(e) $\dfrac{1}{5\sqrt{5}} \approx .089$

(f) $\dfrac{1}{\sqrt{5}} \approx .447$

(g) $\dfrac{7}{12} \approx .583$

(h) $\dfrac{5\sqrt{5}}{12} \approx .931$

(i) $\dfrac{\sqrt{5}}{12} \approx .186$

(j) $\dfrac{5}{12} \approx .417$

Solution: Parametrize C by $\mathbf{r}(t) = (x, y) = (t, t^2)$ for $0 \le t \le 1$. Then $\mathbf{r}'(t) = (1, 2t)$, so $ds = \|\mathbf{r}'(t)\|\,dt = \sqrt{1+4t^2}\,dt$, so

$$\int_C x\,ds = \int_0^1 t\sqrt{1+4t^2}\,dt = \frac{1}{8}\int_0^1 \sqrt{1+u}\,du,$$

after the substitution $u = 4t^2$, $du = 8t\,dt$. But this evaluates to

$$\left.\frac{(1+u)^{3/2}}{12}\right|_0^1 = \frac{5\sqrt{5}-1}{12},$$

and a calculation (or arithmetic with the value $\sqrt{5} \approx 2.236$) gives the estimate. □

12. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F} = x\sin(y)\,\mathbf{i} + xyz\,\mathbf{k},$$

and C is given by the vector function

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}, \qquad 0 \le t \le 1.$$

(a) $\frac{5}{6}$

(b) $\frac{3}{4}$

(c) $\frac{5}{6} - \frac{1}{3}\cos(1)$

(d) $\frac{3}{4} - \frac{1}{3}\cos(1)$

(e) $\frac{5}{6} - \frac{1}{3}\sin(1)$

(f) $\frac{3}{4} - \frac{1}{3}\sin(1)$

(g) $\frac{5}{6} - \frac{1}{2}\sin(1)$

(h) $\frac{3}{4} - \frac{1}{2}\sin(1)$

(i) $\frac{5}{6} - \frac{1}{2}\cos(1)$ $\Leftarrow$

(j) $\frac{3}{4} - \frac{1}{2}\cos(1)$

Solution: $\mathbf{r}'(t) = 1\,\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$, so

$$\mathbf{F} \cdot d\mathbf{r} = \mathbf{F}[\mathbf{r}(\mathbf{t})] \cdot \mathbf{r}'(t)\,dt = (t\sin(t^2) + 3t^8)\,dt.$$

Thus

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (t\sin(t^2) + 3t^8)\,dt \\ &= -\frac{1}{2}\cos(t^2) + 1/3t^9\Big|_0^1 = -\frac{1}{2}\cos(1) + \frac{1}{2} + \frac{1}{3} = \frac{5}{6} - \frac{1}{2}\cos(1). \end{aligned}$$

□

13. A popular Klingon dish requires 0.4 lb of targ tails per warrior. Assume that an average targ tail is 3 inches long and can be parameterized with respect to the arc length by

$$\mathbf{r}(s) = \frac{s}{\sqrt{2}}\,\mathbf{i} + \frac{\cos(s)}{\sqrt{2}}\,\mathbf{j} + \frac{\sin(s)}{\sqrt{2}}\,\mathbf{i}.$$

Assume also that the maximal density of the tail is 0.1 lb/in and that it decreases at the rate of 0.01 lb/in^2 along its length s. What is the minimal number of targ tails needed to prepare this dish for 5 Klingon warriors?

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5
(f) 6
(g) 7
(h) 8 $\Leftarrow$
(i) 9
(j) 10

Solution: Find the mass of an average targ tail by integrating the linear density function $\rho = \rho(s)$. But this satisfies $\rho(0) = 0.1$ and $\rho'(s) = -0.01$, so $\rho(s) = 0.1 - 0.01s$.

Since the parametrization is already with respect to the arc length, the mass integral is just

$$m = \int_{s=0}^{3} \rho(s)\,ds = 0.1s - 0.005s^2\Big|_0^3 = 0.300 - 0.045 = 0.255$$

The number needed is thus (5 warriors)×(0.4 lb/warrior)÷(0.255 lb/tt)=7.8. Round up to 8 for the answer. □

14. If C is the curve given by

$$\mathbf{r}(t) = (1+\sin t)\ \mathbf{i} + \left(1+\sin^2 t\right)\ \mathbf{j} + \left(1+4\sin^3 t\right)\ \mathbf{k}, \qquad 0 \le t \le \frac{\pi}{2},$$

and $\mathbf{F}$ is the radial vector field

$$\mathbf{F}(x,y,z) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k},$$

compute the work done by $\mathbf{F}$ on a particle moving along C.

(a) 0
(b) 3
(c) 15
(d) 30 $\Leftarrow$
(e) 32
(f) 33
(g) 36
(h) 39
(i) 60

Solution: The work done by $\mathbf{F}$ is given by the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

Note that $\mathbf{F} = \nabla f$ for the potential function $f(x,y,z) = x^2 + y^2 + z^2$. Thus $\mathbf{F}$ is conservative and the line integral may be evaluated as $f(C(\frac{\pi}{2})) - f(C(0)) = f(2,2,5) - f(1,1,1) = 30$. $\square$

15. Let $\mathbf{F} = x\,\mathbf{i} + 2y\,\mathbf{j}$ and let $\mathbf{n}$ be the outward unit normal vector to the positively oriented circle $x^2 + y^2 = 16$. Compute the flux integral $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$.

   (a) $3/2$
   (b) $2/3$
   (c) $3\pi/2$
   (d) 3π
   (e) 6π
   (f) 12π
   (g) 24π
   (h) 48π $\Leftarrow$

Solution: Parametrize C by $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j}$. Then $\mathbf{r}'(t) = -4\sin t\,\mathbf{i} + 4\cos t\,\mathbf{j}$, so $\|\mathbf{r}'(t)\| = 4$, so $ds = \|\mathbf{r}'(t)\|\, dt = 4\, dt$.

The outward pointing unit normal to the circle C at $\mathbf{r}(t) \in C$ is $\mathbf{n}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, since $\mathbf{r}(t)$ makes an angle t at the origin with the x-axis.

Thus,

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{n}\, ds &= \int_0^{2\pi} \mathbf{F}[\mathbf{r}(\mathbf{t})] \cdot \mathbf{n}(\mathbf{t}) \|\mathbf{r}'(t)\|\, dt = \int_0^{2\pi} \left[4\cos^2 t + 8\sin^2 t\right] 4\, dt \\ &= 16 \int_0^{2\pi} \left[1 + \sin^2 t\right]\, dt = 16\,[2\pi + \pi] = 48\pi.\end{aligned}$$

Alternatively, let D be the disc of radius 4 enclosed by C and evaluate the flux integral using the divergence theorem:

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}\, dA = \iint_D 3\, dA = 3A(D) = 3 \times \pi(4^2) = 48\pi,$$

since $\operatorname{div} \mathbf{F} = 1 + 2 = 3$. $\square$

16. Determine which of the given sets is open, connected, and simply connected.

I. $\{(x, y) : \ x > 1, y < 2\}$

II. $\{(x, y) : \ 2x^2 + y^2 < 1\}$

III. $\{(x, y) : \ x^2 - 2y^2 < 1\}$

IV. $\{(x, y) : \ 2x^2 - y^2 > 1\}$

V. $\{(x, y) : \ 1 < x^2 + y^2 < 4\}$

(a) I only.

(b) I, II only.

(c) I, II, III only. $\Leftarrow$

(d) I, II, III, IV only.

(e) II, V only.

(f) III, IV only.

(g) All of them.

(h) None of them.

Solution: (I) is a quarter-plane; (II) is the interior of an ellipse; (III) is the region "inside" a hyperbola. These are the only connected regions without holes, as (IV) is the two regions "outside" a hyperbola and (V) is an annulus. $\square$

17. Find a parametrization of the curve $x^{2/3} + y^{2/3} = 4$ and use it to compute the area of the interior.

(a) $3/2$

(b) $2/3$

(c) $3\pi/2$

(d) 3π

(e) 6π

(f) 12π

(g) 24π $\Leftarrow$

(h) 48π

Solution: Let D be enclosed region. Parametrize the bounding curve C with

$$\mathbf{r}(t) = \langle x(t), y(t)\rangle\,; \quad x(t) = 8\cos^3 t, \quad y(t) = 8\sin^3 t; \qquad 0 \le t \le 2\pi,$$

for then $x^{2/3} + y^{2/3} = 4\cos^2 t + 4\sin^2 t = 4$ for all points on $C = \{\mathbf{r}(t) : 0 \le t \le 2\pi\}$.

Let

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = (-y/2)\,\mathbf{i} + (x/2)\,\mathbf{j},$$

for then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ and Green's theorem may be used to compute

$$A(D) = \iint_D 1\,dA = \iint_D 1\,dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt.$$

Now $\mathbf{F}(\mathbf{r}(t)) = \langle -4\sin^3 t, 4\cos^3 t\rangle$ and $\mathbf{r}'(t) = \langle -24\cos^2 t\sin t, 24\sin^2 t\cos t\rangle$, resulting in

$$\begin{aligned} A(D) &= 96\int_0^{2\pi} (\cos^2 t\sin^4 t + \cos^4 t\sin^2 t)\,dt \;=\; 96\int_0^{2\pi} \cos^2 t\sin^2 t\,dt \\ &= 24\int_0^{2\pi} (\sin 2t)^2\,dt \;=\; 24\pi. \end{aligned}$$

□