Math 309.02

Wednesday, Sept. 30 - 12pm

Sec 2.2. INVERSES.

SQUARE MATRICES. ($n \times n$ $\updownarrow$ size n)

* Identity Matrix $I_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$

$$\boxed{I_n A = A \qquad B I_n = B}$$

Def: Given a square matrix (of size n), A,

we say it is INVERTIBLE

if there exists a matrix B (of size n)

such that $AB = BA = I_n$

(Sometimes these are also called NON-SINGULAR

NON-INVERTIBLE = SINGULAR)

If $n = 1$.

$$A = [a].$$

A invertible $\quad a \neq 0$

$$B = \left[\frac{1}{a}\right].$$

Def: Given a square matrix (of size n), A, we say it is INVERTIBLE if there exists a matrix B (of size n) such that $AB = BA = I_n$.

(Sometimes these are also called NON-SINGULAR
NON-INVERTIBLE = SINGULAR)

IN THIS CASE WE WRITE $B = A^{-1}$.

$\underline{n = 2.}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

$D = ad - bc$ (DETERMINANT of A.)

If $D \neq 0$, A is invertible

and $A^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{D} & -\frac{b}{D} \\ -\frac{c}{D} & \frac{a}{D} \end{bmatrix}$.

$$A^{-1}A = \begin{bmatrix} \frac{d}{D} & -\frac{b}{D} \\ -\frac{c}{D} & \frac{a}{d} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \underbrace{\frac{d}{D}\times a - \frac{b}{D}\times c}_{1} & \overbrace{\cancel{\frac{d}{D}\times b - \frac{b}{D}\times d}}^{0} \\ \underbrace{\cancel{-\frac{c}{D}\times a + \frac{a}{D}\times c}}_{0} & \underbrace{-\frac{c}{D} b + \frac{a}{D}\times d}_{1} \end{bmatrix} = I_2.$$

Ex.: Let A be an invertible matrix.

SOLVE: $A\vec{x} = \vec{b}$.

$$\underbrace{A^{-1}A}_{= I_n}\vec{x} = A^{-1}\vec{b} \Rightarrow \boxed{\vec{x} = A^{-1}\vec{b}}$$

Algorithm for inverse

$B = \begin{bmatrix} A & I_n \end{bmatrix}$. This is an $n \times 2n$ matrix.

Attempt to make B into row-reduced echelon form.

If the row reduced echelon form looks like $\begin{bmatrix} I_n & ? \end{bmatrix}$.

then $? = A^{-1}$.

Elementary Matrices.

A square matrix E is called

ELEMENTARY if $E = \begin{bmatrix} * & & & \\ & * & & 0 \\ & & * & \\ 0 & & & * \end{bmatrix}$

$* \neq 0$.

$$E = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$$

Find the inverse.

$$\left[\begin{array}{cc|cc} 1 & a & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & -a \\ 0 & 1 & 0 & 1 \end{array}\right]$$

Elementary Matrices

$$E = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

IN THIS CASE

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & -b+ac \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.$$

Thm: A is invertible

if and only if its columns

are linearly independent.

For any equation $A\vec{x} = \vec{b}$

one has a UNIQUE solution if A invertible.

namely $\vec{x} = A^{-1}\vec{b}$.

If the columns of A are linearly dep. means, there is a non-zero vector $\vec{v}$, such that $A\vec{v} = \vec{0}$.

$\vec{v} = A^{-1}\vec{0}$ if A is invertible

$A = [\vec{u}_1, \ldots, \vec{u}_n]$

$\vec{v} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$.

$A\vec{v} = \vec{0}$

Find $a_1, \ldots, a_n$ not ALL 0

s.t. $a_1\vec{u}_1 + \cdots + a_n\vec{u}_n = \vec{0}$

(30) Find the inverse of

$$\begin{bmatrix} 5 & 10 \\ 4 & 7 \end{bmatrix}.$$

$$\begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & \frac{1}{5} & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & \frac{1}{5} & 0 \\ 0 & -1 & -\frac{4}{5} & 1 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 1 & 2 & \frac{1}{5} & 0 \\ 0 & 1 & \frac{4}{5} & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -7/5 & 2 \\ 0 & 1 & \frac{4}{5} & -1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} -7/5 & 2 \\ \frac{4}{5} & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -7+8 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$