Mathematics 310: Foundations for Higher Mathematics
Model Solutions to the Second Midterm Examination

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No collaboration, notes, texts, problem sets, or worked solutions are permitted. Please write your complete proofs in the blue notebooks. You have until the end of class to answer all of the following problems.

Problem 1: Suppose that $\gcd(a,b)=2$. Prove that $\gcd(3a^2,3b^2)=12$.

Solution 1: Firstly, $2|\gcd(a,b) \implies 2|a \implies 4|a^2 \implies 12|3a^2$. The same reasoning applies to b, so $12|3b^2$, so $12|\gcd(3a^2,3b^2)$.

Secondly, since $\gcd(a,b)=2$ we know that $a_0=a/2$ and $b_0=b/2$ are relatively prime, so by the Unique Factorization Theorem (also by HW5, Problem 6.18) we know that a_0^2 and b_0^2 are relatively prime.

Finally, if $12|d$ and $d>0$ is a common factor of $3a^2$ and $3b^2$, then $d/12$ is a common factor of a_0^2 and b_0^2, so $d=12$. Thus 12 satisfies the definition of greatest common divisor of a^2 and $3b^2$. ■

Problem 2: Determine the last digit (the ones digit) in the base-8 expansion of 2003^{2003}.

Solution 2: $2003 \equiv 3 \pmod 8 \implies 2003^{2003} \equiv 3^{2003} \equiv 3 \pmod 8$. Now $3^2 \equiv 1 \pmod 8$, so

$$3^{2003} \equiv 3^{2\times 1001+1} \equiv (3^2)^{1001}(3^1) \equiv 3 \pmod 8,$$

so the last or ones digit base 8 will be 3. ■

Problem 3: The Golden Ratio is the real number $0<r<1$ satisfying

$$\frac{1}{r}=\frac{r+1}{1}.$$

Prove that the Golden Ratio is not rational.

Solution 3: Since r satisfies the quadratic equation $r^2+r-1=0$, the Rational Roots Theorem implies that r is rational only if $r=\pm 1$. But neither of these two possibilities is a root. ■

Problem 4: Determine the coefficient of xyz in the expansion of $(1+x+y+z+xyz)^9$.

Solution 4: Each term in the expansion will be of the form

$$1^a x^b y^c z^d (xyz)^e = x^{b+e}y^{c+e}z^{d+e}$$

for some choice of nonnegative integer exponents a,b,c,d,e satisfying $a+b+c+d+e=9$. To get xyz, these exponents must also satisfy the equations

$$b+e=1; \qquad c+e=1; \qquad d+e=1.$$

Hence exactly one of the summands can be one in each equation, with the rest being zero. There are just two solutions, which we find by exhaustive search:

(1): $e = 1$; $b = c = d = 0$; $a = 8$.

(2): $b = c = d = 1$; $e = 0$; $a = 6$.

There are $\binom{9}{8,0,0,0,1} = \frac{9!}{8!0!0!0!1!} = 9$ ways to get the solution 1 exponents from 9 factors, and $\binom{9}{6,1,1,1,0} = \frac{9!}{6!1!1!1!0!} = 9 \cdot 8 \cdot 7 = 504$ ways to get the solution 2 exponents. Hence there are a total of $9 + 504 = 513$ ways to get xyz, and the coefficient of xyz is thus 513. ■

Problem 5: For each natural number n, determine with proof the number of permutations of $[n]$ that have no even number as a fixed point.

Solution 5: Let U be the permutations of $[n]$ and for $1 \leq i \leq m \stackrel{\text{def}}{=} \lfloor n/2 \rfloor$ let $A_i \in U$ be those permutations that fix the even element $2i \leq n$. We want to count $N_\emptyset = |U - (A_1 \cup \cdots \cup A_m)|$.

But for every $S \subset [m]$ with $|S| = k$, we have $|\cap_{i \in S} A_i| = (n-k)!$, since the intersection contains all permutations of the $n - k$ elements different from $\{2i : i \in S\}$. There are $\binom{m}{k}$ such subsets S. We can thus count $N_\emptyset$ using the inclusion-exclusion theorem and a regrouping of the sum by the size of S:

$$N_\emptyset = \sum_{S \subset [m]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right| = \sum_{k=0}^{m} (-1)^k (n-k)! \binom{m}{k}.$$

Alternatively, we may break the formula into two parts, depending upon whether $n > 1$ is even or odd:

$$N_\emptyset = \begin{cases} \displaystyle\sum_{k=0}^{m} (-1)^k (2m+1-k)! \binom{m}{k}, & \text{if } n = 2m+1 \text{ is odd;} \\ \displaystyle\sum_{k=0}^{m} (-1)^k (2m-k)! \binom{m}{k}, & \text{if } n = 2m \text{ is even.} \end{cases}$$

In the trivial case $n = 1$, there is exactly one such permutation. ■

Problem 6: In a certain population, only 1% of the individuals have the rare blood type X. A test for this blood type is 99% likely to detect it if the individual has it, but will falsely indicate blood type X in 1% of individuals with another blood type. Suppose a person in this population tests positive for type X. What is the probability that the person's blood type really is X?

Solution 6: First, construct the probability space. Let U be the population and let $A \subseteq U$ be the individuals with blood type X, so that $A^c = U - A$ is the subset of individuals with another blood type. Then $P(A) = 0.01$, $P(A^c) = 0.99$.

Let T be the event that the test indicates blood type X. Then $P(T|A) = 0.99$ and $P(T|A^c) = 0.01$. We want to know $P(A|T)$, which by Bayes' formula is

$$P(A|T) = \frac{P(T|A)P(A)}{P(T|A)P(A) + P(T|A^c)P(A^c)} = \frac{(0.99)(0.01)}{(0.99)(0.01) + (0.01)(0.99)} = \frac{1}{2}.$$

Hence the tested individual has a 50% probability of actually having the rare blood type. ■