Ma 350: Mathematics for Multimedia
Homework Assignment 3

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1. Suppose that $f(t) = 0$ if $|t| \geq 1$, and $|f(t)| < \pi$ for all $t \in (-1,1)$. Prove that $|f_1(t)| < 2\pi$ for all $t \in \mathbf{R}$, where f_1 is the 1-periodization of f.

2. Define the *reflection* R to be the transformation $Ru(t) \stackrel{\text{def}}{=} u(-t)$ acting on the vector space of functions of one real variable. Show that R is a linear transformation, and find a formula for the compositions RF, FR, and RFR, where F is the fraying operator of Equation 3.14.

3. Find an explicit formula by applying the loop operator of Equation 3.23 to the function $u(t) = \cos(t)$, using the interval $I = [0, \pi]$, the rising cutoff function $r(t) = r_0(t)$ defined in Equation 3.12, and the reach $\epsilon = \pi/2$. (Note: the result should not be a continuous function.)

4. Show that the following operators can be used for fraying and splicing:

$$
\begin{aligned}
Fu(t) &= \begin{cases} [u(t) + u(-t)]/\sqrt{2}, & \text{if } 0 < t < 1, \\ [u(t) - u(-t)]/\sqrt{2}, & \text{if } -1 < t < 0, \\ u(t), & \text{otherwise;} \end{cases} \\
Su(t) &= \begin{cases} [u(t) - u(-t)]/\sqrt{2}, & \text{if } 0 < t < 1, \\ [u(t) + u(-t)]/\sqrt{2}, & \text{if } -1 < t < 0, \\ u(t), & \text{otherwise.} \end{cases}
\end{aligned}
$$

   That is, show that

   (i) $Su(t) = u(t)$ and $Fu(t) = u(t)$ if $|t| > 1$.

   (ii) S and F are linear transformations of functions on $\mathbf{R}$.

   (iii) S and F are inverses.

   (iv) If u belongs to $C^d(\mathbf{R})$, then for $0 \leq n \leq d$,

$$[Fu]^{(n)}(0+) = 0 \quad \text{for odd } n; \qquad [Fu]^{(n)}(0-) = 0 \quad \text{for even } n.$$

(v) If u belongs to $C^d(\mathbf{R} \setminus \{0\})$ and has one-sided limits $u^{(n)}(0\pm)$ for $0 \le n \le d$ which satisfy

$$u^{(n)}(0-) = 0 \quad \text{for odd } n; \qquad u^{(n)}(0+) = 0 \quad \text{for even } n,$$

then defining $Su(0) = Su(0+)$ yields $Su \in C^d(-1, 1)$.

5. Suppose that r_1 and r_2 are rising cutoff functions, namely they each satisfy Equation 3.11. Prove that $r_1(2|r_2(t)|^2 - 1)$, the composition of r_1 with $2|r_2|^2 - 1$, also satisfies that equation. Under what additional assumptions will the composition function be smooth?

6. Show that the set of functions $\{\sqrt{2} \sin \pi n t : n = 1, 2, \ldots\}$ is orthonormal with respect to the inner product

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int_0^1 f(t) g(t)\, dt.$$

That is, show that

$$\left\langle \sqrt{2} \sin \pi n t, \sqrt{2} \sin \pi m t \right\rangle = \begin{cases} 0, & \text{if } n \neq m, \\ 1, & \text{if } n = m, \end{cases}$$

for all $n, m \in \mathbf{Z}^+$.

7. Compute the sine-cosine Fourier series of the 1-periodic function $f(x) = \sin^2(\pi x)$. (Hint: use a trigonometric identity.)

8. Compute the complex exponential Fourier series of the 1-periodic function $\cos(2\pi k t - d)$, where $k > 0$ is an integer and d is a constant real number.

9. Show that if $|c(n)| < 1/|n|^4$ for all integers $n \neq 0$, then the 1-periodic function $f = f(t)$ which is the inverse Fourier transform of the sequence $\{c(n)\}$ must have a continuous derivative at every $t \in [0, 1]$.

10. Show that the functions $\phi_j(t) = \mathbf{1}(2^j t - 1)$, $j \in \mathbf{Z}^+$, are an orthogonal collection with respect to the inner product

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int_0^1 \overline{f(t)} g(t)\, dt.$$

Here $\mathbf{1} = \mathbf{1}_{[0,1)}$ is the indicator function of the interval $[0, 1)$. How can this collection be made orthonormal? What is the linear span of $\{\phi_j : j \in \mathbf{Z}^+\}$?

11. Suppose that ϕ has Fourier integral transform $\mathcal{F}\phi$. Let $\phi_k(x) \stackrel{\text{def}}{=} \phi(x + k)$. Show that $\mathcal{F}\phi_k(\xi) = e^{2\pi i k \xi} \mathcal{F}\phi(\xi)$.

12. Suppose that ϕ has Fourier integral transform $\mathcal{F}\phi$. Let $\phi_a(x) \stackrel{\text{def}}{=} \phi(ax)$, for $a > 0$. Show that $\mathcal{F}\phi_a(\xi) = \frac{1}{a} \mathcal{F}\phi(\xi/a)$.

13. Compute the inverse Fourier integral transform of the function

$$\psi(\xi) = \begin{cases} 1, & \text{if } -2 \le \xi < -1 \text{ or } 1 < \xi \le 2; \\ 0, & \text{otherwise.} \end{cases}$$

(Hint: notice that $\psi(\xi) = \mathbf{1}_I(\xi/4) - \mathbf{1}_I(\xi/2)$, where $I = [-\frac{1}{2}, \frac{1}{2}]$.)

14. Compute the Fourier integral transform of the bump function

$$b(x) = \begin{cases} 2 - 2|x|, & \text{if } -1 \le x < -\frac{1}{2} \text{ or } \frac{1}{2} < x \le 1; \\ 1, & \text{if } -\frac{1}{2} \le x \le \frac{1}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

15. Show that the vectors $\bar{\omega}_n \in \mathbf{C}^N$, $n = 0, 1, \ldots, N-1$ defined by $\bar{\omega}_n(k) = \exp(-2\pi ink/N)$ form an orthonormal basis with respect to the inner product

$$\langle f, g \rangle \stackrel{\text{def}}{=} \frac{1}{N} \sum_{k=0}^{N-1} \overline{f(k)}\, g(k).$$

16. Write out explicitly the matrices for the 3×3 discrete inverse Fourier and Hartley transforms (F_3^{-1} and H_3^{-1}).

17. What is the matrix $(C_N^{IV})^2$ of the square of $N \times N$ DCT-IV? Give a formula for every positive integer N.

18. What is the matrix $(S_N^{IV})^4$ of the fourth power of $N \times N$ DST-IV? Give a formula for every positive integer N.

19. Write out explicitly the matrices C_3^{IV} and S_3^{IV} used for the 3×3 DCT-IV and DST-IV, respectively.

20. Show that $\operatorname{cas}(A)\operatorname{cas}(B) = \cos(A - B) + \sin(A + B)$.

21. Prove that the $N \times N$ discrete Hartley transform matrix $\sqrt{\frac{1}{N}}\, H_N$ is symmetric and unitary. ($H_N(m, n) = \operatorname{cas}(2\pi mn/N) = \cos(2\pi mn/N) + \sin(2\pi mn/N)$ is defined in Equation 3.54.)