Ma 350: Mathematics for Multimedia
Homework Assignment 4

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1. Find the Lagrange polynomial through the points $(1, 2)$, $(2, 5)$, $(3, 4)$.

2. Fix x and find a formula for the value $y = f(x)$ of the Lagrange polynomial f through the points $(-1, p)$, $(0, q)$, and $(1, r)$, in terms of x, p, q, and r. Then find d^2y/dx^2.

3. Suppose $f(x) = x^k$ for a fixed integer $k > 0$. Prove that $\Delta^k f(k) = k!$. (Hint: consider c_k in Equation 4.7.)

4. For all $N \in \{0, 1, \ldots, 10\}$, find the expansion of the function $f(x) = x^N$ in Chebyshev polynomials $T_0(x), T_1(x), \ldots, T_N(x)$.

5. Suppose that $g(x) = mx + b$ for some constants m, b. Fix $n > 0$ and a grid $x_0 < \cdots < x_n$ and let f be the piecewise constant approximation to f on this grid. Show that
$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_n} g(x)\,dx.$$

6. Suppose that `x[]` and `y[]` are both increasing arrays of real numbers indexed by $0, 1, \ldots, N$. Implement `pwlinverse(x[],y[],y0,N)`, a function that returns the value x_0 satisfying $f(x_0) = y_0$, where f is the piecewise linear function interpolating the set $\{(x_k, y_k) : k = 0, 1, \ldots, N\} \subset \mathbf{R}^2$.

7. Suppose that $x_1 < x_2$, and let y_1, y_2 be arbitrary real numbers. Let f be the linear function interpolating $\{(x_1, y_1), (x_2, y_2)\}$. On what interval (if any) is $f > 0$? On what interval (if any) is $f < 0$?

8. Let $f(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\frac{x_1^2(y_3-y_2)+x_2^2(y_1-y_3)+x_3^2(y_2-y_1)}{x_1(y_3-y_2)+x_2(y_1-y_3)+x_3(y_2-y_1)}$ as in Equation 4.16, where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$ and $\mathbf{y} = (y_1, y_2, y_3) \in \mathbf{R}^3$. For $\mathbf{a} = (a, a, a) \in \mathbf{R}^3$, prove that $f(\mathbf{x}, \mathbf{y} + \mathbf{a}) = f(\mathbf{x}, \mathbf{y})$ and that $f(\mathbf{x} + \mathbf{a}, \mathbf{y}) = f(\mathbf{x}, \mathbf{y}) + a$.

9. Let $\mathbf{B} \stackrel{\text{def}}{=} \{\phi_n : n \in \mathbf{Z}\} \subset L^2(\mathbf{R})$ be the orthonormal subset of functions of one real variable $\phi_n(x) \stackrel{\text{def}}{=} \mathbf{1}_{[n,n+1)}$. Determine with proof whether $\operatorname{span}\mathbf{B} = \overline{\operatorname{span}}\,\mathbf{B}$.

10. Suppose that $\phi(x)$ is the following step function:
$$\phi(x) = \mathbf{1}_{(-\frac{1}{2},\frac{1}{2})} \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } |x| < \frac{1}{2}; \\ 0, & \text{if } x \le -\frac{1}{2} \text{ or } x \ge \frac{1}{2}, \end{cases}$$
Let $c = \{c(n) : n \in \mathbf{Z}\}$ be a sequence of complex numbers, and put $f = f(t) = \sum_{n\in\mathbf{Z}} c(n)\phi(t-n)$. Show that $f \in L^2(\mathbf{R})$ if and only if $c \in \ell^2$. (Hint: compute $\|f\|^2$ in terms of $\{|c(n)|^2\}$.)