# Ma 350: Mathematics for Multimedia
# Homework Assignment 5

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1. Draw the graphs of $w(\frac{t}{3} - 4)$ and $w(\frac{t-4}{3})$ on one set of axes for the Haar function $w(t)$ defined in Equation 5.2.

2. For each $(a, b) \in \mathbf{Aff}$, define the linear operator $\zeta(a, b) : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ by

$$\zeta(a, b)f(t) \stackrel{\text{def}}{=} \sqrt{a}\, f(at + b).$$

   Is $\zeta$ a representation of the group $\mathbf{Aff}$? If so, is it a faithful representation? Is it unitary?

3. Let $f = f(\mathbf{a}) = f(a, b)$ be the function on $\mathbf{Aff}$ defined by $f(\mathbf{a}) = \mathbf{1}_D(\mathbf{a})$, where $\mathbf{1}_D$ is the indicator function of the region $D = \{\mathbf{a} = (a, b) : A < a < A', B < b < B'\} \subset \mathbf{Aff}$ for $0 < A < A'$ and $-\infty < B < B' < \infty$. Evaluate $\int_{\mathbf{Aff}} f(\mathbf{a})\, d\mathbf{a}$ using the normalized left-invariant integral on $\mathbf{Aff}$.

4. Let $w = w(t)$ be the Haar mother function and define

$$\phi^J_{M,K}(t) \stackrel{\text{def}}{=} \sum_{j=M+1}^{M+J} \frac{1}{2^j} w\left(\frac{t-K}{2^j}\right)$$

   for arbitrary fixed $K \in \mathbf{R}$ and $M, J \in \mathbf{Z}$ with $J \geq 0$. Prove that

$$\lim_{J\to\infty} \phi^J_{M,K}(t) = 2^{-M}\mathbf{1}_{[K,K+2^M)}(t) \stackrel{\text{def}}{=} \phi_{M,K}(t),$$

   and also that $\left\langle \phi^J_{M,K}, u \right\rangle \to \langle \phi_{M,K}, u\rangle$ as $J \to \infty$ for any function $u \in L^2(\mathbf{R})$.

5. Let $d$ be a positive integer. Show that the following function has $d - 1$ continuous derivatives:

$$\phi(\xi) = \begin{cases} \xi^d e^{-\xi^2}, & \text{if } \xi > 0; \\ 0, & \text{if } \xi \leq 0. \end{cases}$$

6. Fix an integer $d > 0$ and define

$$\phi(\xi) = \begin{cases} \exp\left[-(\log\xi)^{2d}\right], & \text{if } \xi > 0; \\ 0, & \text{if } \xi \le 0. \end{cases}$$

Show that for any $n, m \in \mathbf{N}$, we have $\phi^{(n)}(\xi) = O(1/|\xi|^m)$ as $\xi \to \pm\infty$.

7. Let $u = u(x)$ be a continuous function in $L^2(\mathbf{R})$. Let $w = w(t)$ be the Haar mother function defined in Equation 5.2. Compute the derivatives with respect to $a$ and $b$ of the wavelet transform $Wu = Wu(a, b)$ using mother function $w$.

8. Compute $\|w\|$, where

$$\mathcal{F}w(\xi) = \begin{cases} e^{-(\log\xi)^2}, & \text{if } \xi > 0; \\ 0, & \text{if } \xi \le 0. \end{cases}$$

(Hint: use Plancherel's theorem and Equation B.6 in Appendix B.)

9. Let $w$ be the function defined by

$$\mathcal{F}w(\xi) = \begin{cases} e^{-(\log|\xi|)^2}, & \text{if } \xi \neq 0; \\ 0, & \text{if } \xi = 0. \end{cases}$$

Show that $w$ is admissible and compute its normalization constant $c_w$.

10. Fix $A < 0$, $B > 0$, and $R > 1$ and suppose that $w = w(x)$ is a function satisfying $\mathcal{F}w(\xi) = 1$ if $RA < \xi < A$ or $B < \xi < RB$, with $\mathcal{F}w(\xi) = 0$, otherwise. Show that $w$ satisfies the admissibility condition of Theorem 5.2, and compute the normalization constant $c_w$. Give a formula for $w$.