Ma 350: Mathematics for Multimedia
Homework Assignment 6

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1. Let w be the Haar mother function defined by Equation 5.2. Prove that the set of functions $\{\psi_k : k \in \mathbf{Z}\}$ defined by $\psi_k(t) = w(t-k)$ is orthonormal.

2. Show that if $h = \{h(k) : k \in \mathbf{Z}\}$ is a self-orthonormal filter, and M is any fixed integer, then the sequence defined by

$$g(k) = (-1)^k \overline{h(2M-1-k)}, \qquad \text{for all } k \in \mathbf{Z},$$

satisfies the completeness condition of Equation 5.45.

3. a. Are there any real-valued orthogonal low-pass CQFs of length 4 satisfying the antisymmetry condition $h(0) = -h(3)$ and $h(1) = -h(2)$?

   b. Are there any real-valued orthogonal low-pass CQFs of length 4 satisfying the symmetry condition $h(0) = h(3)$ and $h(1) = h(2)$?

4. Suppose that an orthogonal MRA has a scaling function ϕ satisfying $\phi(t) = 0$ for $t \notin [a,b]$. Prove that the low-pass filter h for this MRA must satisfy $h(n) = 0$ for all $n \notin [2a-b, 2b-a]$. (This makes explicit the finite support of h in Equation 5.36.)

5. Suppose that x, y, a, b are integers with $x \geq y$ and $b \geq a$. Let $u = \{u(k) : k \in \mathbf{Z}\}$ be a sequence supported in $[x, y]$ and let $f = \{f(k) : k \in \mathbf{Z}\}$ be a filter sequence supported in $[a, b]$ that defines a filter transform F and its adjoint F^* as in Equations 5.61 and 5.62. What is the support interval for FF^*u? What if f satisfies the self-orthonormality condition?

6. Suppose that $h = \{h(k) : k \in \mathbf{Z}\}$ and $g = \{g(k) : k \in \mathbf{Z}\}$ satisfy the orthogonal CQF conditions. Show that the 2-periodizations h_2, g_2 of h and g are the Haar filters. Namely, show that $h_2(0) = h_2(1) = g_2(0) = -g_2(1) = 1/\sqrt{2}$.

7. Let ϕ be the scaling function of an orthogonal MRA, and let ψ be the associated mother function. For $(x, y) \in \mathbf{R}^2$, define

$$e_0(x,y) = \phi(x)\phi(y), \qquad e_1(x,y) = \phi(x)\psi(y)$$
$$e_2(x,y) = \psi(x)\phi(y), \qquad e_3(x,y) = \psi(x)\psi(y).$$

Prove that the functions $\{e_n : n = 0, 1, 2, 3\}$ are orthonormal in $L^2(\mathbf{R}^2)$, the inner product space of square-integrable functions on $\mathbf{R}^2$.

8. Fix an integer $N > 1$ and consider a graph with vertices labeled $1, \ldots, N$ with each pair of vertices connected by an edge. Compute the total number of edges, and list them.

9. Construct a prefix code for the alphabet $A = \{a, b, c, d\}$ with codeword lengths 1,2,3,4, or prove that none exists.

10. Construct a prefix code for the 26-letter English alphabet $A = \{a, b, c, \ldots, z\}$ with longest codeword 4, or prove that none exists.

11. Suppose we have two prefix codes, $\mathbf{c}_0(a, b) = (1, 0)$ and $\mathbf{c}_1(a, b) = (0, 1)$, for the alphabet $A = \{a, b\}$. Show that the following *dynamic encoding* is uniquely decipherable by finding a decoding algorithm:

Simple Dynamic Encoding Example

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dynamicencoding0( msg[], M ):
[0] Initialize n=0
[1] For m=1 to M, do [2] to [3]
[2]    Transmit msg[m] using code n
[3]    If msg[m]=='b', then toggle n = 1-n
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(This encoding is called dynamic because the codeword for a letter might change as a message is encoded, in contrast with the *static encodings* studied in this chapter. It gives an example of a uniquely decipherable and instantaneous code which is nevertheless not a prefix code.)