Mathematics 411: Advanced Calculus I
Problem Set 1 — due Thursday, September 6, 2001

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Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions.

Problem 1: Prove that if $n \in \mathbf{Z}^+$, then

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k}.$$

Problem 2: Prove, using induction, that every nonempty subset of $\mathbf{Z}^+$ contains a smallest element. This is called the *well-ordering principle.*

Problem 3: Find the rational number whose decimal expansion is $0.111234\ldots$, where the three digits '234' repeat indefinitely thereafter.

Problem 4: Prove that $\sqrt{3} - \sqrt{2}$ is irrational.

Problem 5: Prove that between any rational number x and irrational number $y > x$ there is both an irrational number $y' \neq y$ and a rational number $x' \neq x$.

Problem 6: (a) Suppose that A and B are nonempty subsets of $\mathbf{R}^+$ which are bounded above with $a = \sup A$ and $b = \sup B$. For $A \circ B \overset{\text{def}}{=} \{x^2 + y^2 : x \in A, y \in B\}$, show that $\sup A \circ B = a^2 + b^2$.

(b) Find two subsets A and B of $\mathbf{R}$ which are bounded above but for which $A \circ B$ is **not** bounded above.

Problem 7: Prove the triangle inequality $\|a + b\| \leq \|a\| + \|b\|$ for n-component vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$:

$$\sqrt{\sum_{k=1}^{n}(a_k + b_k)^2} \leq \sqrt{\sum_{k=1}^{n} a_k^2} + \sqrt{\sum_{k=1}^{n} b_k^2}.$$

Problem 8: If $z = x + iy$ for real x, y, define the *complex conjugate* of z by $\bar{z} \overset{\text{def}}{=} x - iy$. Prove that $\overline{z + w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$, $z\bar{z} = |z|^2$, $z + \bar{z} = 2\mathrm{Re}\, z$, and $z - \bar{z} = 2i\mathrm{Im}\, z$.

Problem 9: Sketch the following subsets of $\mathbf{C}$: $|z| = 1$, $|z| < 1$, $z + \bar{z} = 1$, $z - \bar{z} = 12$, and $z + \bar{z} = |z|^2$.

Problem 10: Prove that the n n^{th} roots of 1 are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$, where $\alpha = e^{2\pi i/n}$, and that α, α^2 through α^{n-1} each solve the equation $1 + z + \cdots + z^{n-1} = 0$. (Hint: use Problem 1).