Mathematics 411: Advanced Calculus I
Problem Set 2 — Due Thursday, September 20, 2001

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Please return your solutions to me by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

Put $(a,b) = \{\{a\},\{a,b\}\}$ for problems 1 and 2.

Problem 1: Prove that $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$.

Problem 2: Define an "ordered n-tuple" $(a_1, a_2, \ldots, a_n)$ inductively for $n > 2$ by the formula $(a_1, a_2, \ldots, a_n) \stackrel{\text{def}}{=} ((a_1, a_2, \ldots, a_{n-1}), a_n)$. Prove that $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$ for all $i = 1, 2, \ldots, n$.

For problems 3 and 4, define an *equivalence relation* S to be a relation with the following three properties:

reflexivity: $a \in \operatorname{Dom} S \implies (a,a) \in S$;

symmetry: $(a,b) \in S \implies (b,a) \in S$;

transitivity: If $(a,b) \in S$ and $(b,c) \in S$ then $(a,c) \in S$.

Such a relation generalizes "=" and if $(a,b) \in S$ then we say that "a and b are equivalent with respect to S."

Problem 3: Determine which of the following plane relations are equivalence relations: (a) $S = \{(x,y) : x^2 = y^2\}$; (b) $S = \{(x,y) : x^2 + y^2 < 1\}$; (c) $S = \{(x,y) : xy \geq 0\}$.

Problem 4: Fix $p \in \mathbf{Z}^+$ and let $S = \{(x,y) \in \mathbf{Z}^+ \times \mathbf{Z}^+ : p|(x-y)\}$. Show that S is an equivalence relation. (If $(x,y) \in S$, then we say that x and y are *congruent modulo* p and write $x \equiv y \pmod{p}$.)

For problems 5, 6, 7 and 8, let $f : S \to T$ be a function and for each $Y \subset T$ define $f^{-1}(Y) \stackrel{\text{def}}{=} \{x \in S : f(x) \in Y\}$.

Problem 5: Prove that $X \subset f^{-1}[f(X)]$ for any $X \subset S$.

Problem 6: Prove that $f\left[f^{-1}(Y)\right] \subset Y$ for any $Y \subset T$.

Problem 7: Prove that $f\left[f^{-1}(Y)\right] = Y$ for any $Y \subset T$ if and only if $f(S) = T$.

Problem 8: Prove that the following five statements are equivalent:

(a) f is one-to-one on S.

(b) $f(A \cap B) = f(A) \cap f(B)$ for all subsets A and B of S.

(c) $f^{-1}[f(A)] = A$ for every subset A of S.

(d) If $A \subset S$, $B \subset S$, and $A \cap B = \emptyset$, then $f(A) \cap f(B) = \emptyset$.

(e) If $A \subset S$, $B \subset S$, and $A \subset B$, then $f(B - A) = f(B) - f(A)$.

(Hint: Suppose one could show that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (d) $\Longrightarrow$ (e) $\Longrightarrow$ (c) $\Longrightarrow$ (a) $\ldots$)

Problem 9: Suppose that A is countable. Prove that if B is uncountable, then $B - A$ is uncountable.

Problem 10: Prove that every uncountable set contains a countably infinite subset.