

**Mathematics 411: Advanced Calculus I**  
**Problem Set 5 — Due Thursday, November 1, 2001**

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Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions.

**Problem 1:** Prove that a metric space  $S$  is disconnected if and only if there is a subset  $A \subset S$  which is neither empty nor all of  $S$ , but which is both open and closed in  $S$ .

**Problem 2:** A set  $S \subset \mathbf{R}^n$  is called *starlike* if there is some *base point*  $x \in S$  such that for every point  $y \in S$ , the line between  $x$  and  $y$  is contained in  $S$ .

- (a) Prove that every convex subset of  $\mathbf{R}^n$  is starlike.
- (b) Prove that every starlike subset of  $\mathbf{R}^n$  is connected.

**Problem 3:** Prove that if  $f : \mathbf{R} \rightarrow \mathbf{R}$  is continuous and one-to-one on a compact interval  $[a, b]$ , then  $f$  must be strictly monotonic on  $[a, b]$ .

**Problem 4:** A function  $f : \mathbf{R} \rightarrow \mathbf{R}$  is said to satisfy a *Lipschitz condition of order  $\alpha$*  at a point  $c$  in its domain if  $(\exists M > 0)(\exists r > 0)(\forall x \in B(c; r), x \neq c) |f(x) - f(c)| < M|x - c|^\alpha$ .

- (a) Prove that if  $f$  satisfies a Lipschitz condition of order  $\alpha > 0$  at  $c$ , then  $f$  is continuous at  $c$ .
- (b) Prove that if  $f$  satisfies a Lipschitz condition of order  $\alpha > 1$  at  $c$ , then  $f$  is differentiable at  $c$ .
- (c) Find a function  $f$  satisfying a Lipschitz condition of order  $\alpha = 1$  at  $c$  but which is not differentiable at  $c$ .

**Problem 5:** Suppose that  $f$  is defined on  $(0, 1]$  and has a bounded derivative in  $(0, 1)$  (*i.e.*, there is a finite  $M > 0$  such that  $|f'(x)| \leq M$  for all  $x \in (0, 1)$ ). Put  $a_n = f(1/n)$  for  $n = 1, 2, 3, \dots$ . Prove that  $\lim_{n \rightarrow \infty} a_n$  exists. (Hint: use the Cauchy criterion.)

**Problem 6:** Let  $f$  be continuous on  $[0, 1]$  with  $f(0) = 0$  and  $f'(x)$  finite at each  $x \in (0, 1)$ . Suppose  $f'(x)$  is an increasing function on  $(0, 1)$ . Prove that  $g(x) \stackrel{\text{def}}{=} f(x)/x$  is an increasing function on  $(0, 1)$ .

**Problem 7:** Prove that if  $f$  has a finite third derivative  $f'''$  in  $(a, b)$ , and  $f(a) = f(b) = f'(a) = f'(b) = 0$ , then there must be some point  $c \in (a, b)$  for which  $f'''(c) = 0$ .

**Problem 8:** Suppose that the vector-valued function  $\mathbf{f}$  is differentiable at each point  $t \in (a, b)$ , and that  $\|\mathbf{f}\|$  is constant on  $(a, b)$ . Prove that  $\mathbf{f}(t) \cdot \mathbf{f}'(t) = 0$  for all  $t \in (a, b)$ .

**Problem 9:** Define a real-valued function  $f$  of two real variables as follows:

$$f(x, y) = \frac{xy}{x^2 + y^2}, \quad (x, y) \neq 0; \quad f(0, 0) = 0.$$

- (a) Prove that the partial derivatives  $D_1 f(x, y)$  and  $D_2 f(x, y)$  exist for every  $(x, y) \in \mathbf{R}^2$  and find explicit formulas for them.
- (b) Show that  $f$  is not continuous at  $(0, 0)$ .

**Problem 10:** Let  $S$  be an open set in  $\mathbf{C}$  and let  $S^*$  be the set of complex conjugates of points of  $S$ :  $S^* \stackrel{\text{def}}{=} \{\bar{z} : z \in S\}$ . If  $f$  is defined on  $S$ , define  $g$  on  $S^*$  by the formula  $g(z) = \overline{f(\bar{z})}$ . Prove that if  $f$  is differentiable at  $c \in \text{int}S$ , then  $g$  is differentiable at  $\bar{c}$ .