Mathematics 411: Advanced Calculus I
Problem Set 6 — Due Tuesday, November 15, 2001

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Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions.

Problem 1: Determine (with proof) whether the function $f(x) \stackrel{\text{def}}{=} x^{1/5}\cos(\pi/2x)$ if $x \neq 0$, with $f(0) \stackrel{\text{def}}{=} 0$, has bounded variation on the interval $[-1, 1]$.

Problem 2: A function $\mu = \mu(x)$ defined on $\mathbf{R}^+$ is called a *Marcinkiewicz multiplier* if there is some $M < \infty$ such that $V_\mu(2^j, 2^{j+1}) < M$ for all integers j; that is, μ has uniformly bounded variation on intervals of the form $[2^j, 2^{j+1}]$.

(a) Prove that $\mu(x) = \log x$ is a Marcinkiewicz multiplier. Thus such functions do not have to be bounded.

(b) Prove that μ is a Marcinkiewicz multiplier if and only if there is some $\lambda > 1$ and some $N < \infty$ such that $V_\mu(a, \lambda a) < N$ for every $a > 0$.

For Problems 3 and 4, a real-valued function f defined on $[a, b] \subset \mathbf{R}$ is said to *absolutely continuous* on $[a, b]$ if for every $\epsilon > 0$ there is $\delta > 0$ such that for every finite collection of disjoint open subintervals $(a_i, b_i) \subset [a, b]$ with $\sum_i |b_i - a_i| < \delta$, we have $\sum_i |f(b_i) - f(a_i)| < \epsilon$.

Problem 3: Prove that a function which is absolutely continuous on $[a, b]$ is continuous and of bounded variation on $[a, b]$.

Problem 4: Prove that if f and g are absolutely continuous, then so are $f + g$, $f - g$, cg (where c is a constant), $|f|$, and fg. Prove also that if in addition $|g(x)| \geq \eta > 0$ for all x, then f/g is absolutely continuous.

Problem 5: Suppose that $\mathbf{f}$ is a rectifiable path of length L defined on $[a, b]$ and assume that $\mathbf{f}$ is not constant on any subinterval of $[a, b]$. Let $s(x) = \Lambda_{\mathbf{f}}(a, x)$ if $a < x \leq b$ and put $s(a) = 0$. Prove that s^{-1} exists and is continuous on $[0, L]$.