Mathematics 411: Advanced Calculus I
Worked Solutions to Problem Set 2

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Put $(a,b) = \{\{a\},\{a,b\}\}$ for problems 1 and 2.

Problem 1: Prove that $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$.

Solution 1: For $\Longleftarrow$, it is evident that $(a,b) = (c,d)$ if $a = c$ and $b = d$. For $\Longrightarrow$, we note that $(a,b) = (c,d) \implies (\{a\} = \{c\}$ or $\{a\} = \{c,d\})$. In the latter case, we must have $a = c = d$, so in either case we have $a = c$. But also, $\{a,b\} = \{c,d\}$ or $\{a,b\} = \{c\}$, so either $b = d$ or else $b = c = a$, and since $\{c,d\} = \{a,b\}$ or $\{c,d\} = \{a\}$ we may then conclude that $d = a = c = b$. ■

Problem 2: Define an "ordered n-tuple" $(a_1, a_2, \ldots, a_n)$ inductively for $n > 2$ by the formula $(a_1, a_2, \ldots, a_n) \stackrel{\text{def}}{=} ((a_1, a_2, \ldots, a_{n-1}), a_n)$. Prove that $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$ for all $i = 1, 2, \ldots, n$.

Solution 2: The case $n = 2$ is covered by solution 1 so we consider only $n > 2$. Again, the $\Longleftarrow$ direction is evident. For $\Longrightarrow$, suppose that $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ and that the result holds for all $(n-1)$-tuples. Then $(a_1, a_2, \ldots, a_{n-1}) = (b_1, b_2, \ldots, b_{n-1})$ so $a_i = b_i$ for $1 \leq i \leq n-1$ by the inductive hypothesis, while $a_n = b_n$ because the second members of the outer ordered pairs must be equal. ■

For problems 3 and 4, define an *equivalence relation* S to be a relation with the following three properties:

reflexivity: $a \in \operatorname{Dom} S \implies (a,a) \in S$;

symmetry: $(a,b) \in S \implies (b,a) \in S$;

transitivity: If $(a,b) \in S$ and $(b,c) \in S$ then $(a,c) \in S$.

Such a relation generalizes "=" and if $(a, b) \in S$ then we say that "a and b are equivalent with respect to S."

Problem 3: Determine which of the following plane relations are equivalence relations: (a) $S = \{(x, y) : x^2 = y^2\}$; (b) $S = \{(x, y) : x^2 + y^2 < 1$; (c) $S = \{(x, y) : xy \geq 0\}$.

Solution 3: (a) Yes, since it is equivalent to $|x| = |y|$, which is transitive, reflexive and symmetric.

(b) No, since $0.9 \in \operatorname{Dom} S$ but $(0.9, 0.9) \notin S$.

(c) No, since $(-1)(0) = 0 \geq 0$, and $(0)(1) = 0 \geq 0$, but $(-1)(1) = -1 < 0$, so the relation is not transitive. ■

Problem 4: Fix $p \in \mathbf{Z}^+$ and let $S = \{(x, y) \in \mathbf{Z}^+ \times \mathbf{Z}^+ : p|(x - y)\}$. Show that S is an equivalence relation. (If $(x, y) \in S$, then we say that x and y are *congruent modulo* p and write $x \equiv y \pmod{p}$.)

Solution 4: S is reflexive: $p|(x - y) \iff p|(y - x)$. S is symmetric: $p|(x - x)$ since $x - x = 0 = 0p$. S is transitive: if $p|(x - y)$ and $p|(y - z)$ then $p|\left[(x - y) + (y - z)\right]$ so $p|(x - z)$. ■

For problems 5, 6, 7 and 8, let $f : S \to T$ be a function and for each $Y \subset T$ define $f^{-1}(Y) \stackrel{\text{def}}{=} \{x \in S : f(x) \in Y\}$.

Problem 5: Prove that $X \subset f^{-1}\left[f(X)\right]$ for any $X \subset S$.

Solution 5: If $x \in X$ then $f(x) \in f(X)$ so $x \in f^{-1}\left[f(X)\right]$ by the definition with $Y = f(X)$. ■

Problem 6: Prove that $f\left[f^{-1}(Y)\right] \subset Y$ for any $Y \subset T$.

Solution 6: By the definition, for every $x \in f^{-1}(Y)$ we have $f(x) \in Y$. Thus $f\left[f^{-1}(Y)\right] \subset Y$. ■

Problem 7: Prove that $f\left[f^{-1}(Y)\right] = Y$ for any $Y \subset T$ if and only if $f(S) = T$.

Solution 7: For $\Longrightarrow$, just take $Y = T$. Then $f\left[f^{-1}(T)\right] = T \implies T \subset \operatorname{Ran} f$. But since $\operatorname{Ran} f \subset T$ we conclude $\operatorname{Ran} f = T$, and since $S = \operatorname{Dom} f$ and $f(\operatorname{Dom} f) = \operatorname{Ran} f$ we know that $f(S) = T$.

For $\Longleftarrow$, suppose that $f\left[f^{-1}(Y)\right] \neq Y$ for some $Y \subset T$. By solution 6, $f\left[f^{-1}(Y)\right]$ must be strictly smaller than Y so there must be some $y \in Y$ with $y \notin f\left[f^{-1}(Y)\right]$. But then there can be no $x \in S$ with $f(x) = y$, since if there were we would have $x \in f^{-1}(Y) \implies y = f(x) \in f\left[f^{-1}(Y)\right]$. Hence $f(S)$ omits at least y and thus cannot be all of T. ■

Problem 8: Prove that the following five statements are equivalent:

(a) f is one-to-one on S.

(b) $f(A \cap B) = f(A) \cap f(B)$ for all subsets A and B of S.

(c) $f^{-1}\left[f(A)\right] = A$ for every subset A of S.

(d) If $A \subset S$, $B \subset S$, and $A \cap B = \emptyset$, then $f(A) \cap f(B) = \emptyset$.

(e) If $A \subset S$, $B \subset S$, and $A \subset B$, then $f(B - A) = f(B) - f(A)$.

Solution 8: We first remark that $f(X) = \emptyset \iff X = \emptyset$. Now we show in steps that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (d) $\Longrightarrow$ (e) $\Longrightarrow$ (c) $\Longrightarrow$ (a). Such a closed loop of implications shows that all the statements are equivalent:

(a) $\Longrightarrow$ (b): Suppose $y \in f(A) \cap f(B)$. Then there exists $a \in A$ and $b \in B$ with $f(a) = y = f(b)$. But f is 1-1 so this means $a = b$ and both belong to $A \cap B$. Thus $y = f(a) \in f(A \cap B)$ and we have shown that $f(A) \cap f(B) \subset f(A \cap B)$. At the same time, $A \cap B \subset A$ so $f(A \cap B) \subset f(A)$ and $A \cap B \subset B$ so $f(A \cap B) \subset f(B)$, and therefore $f(A \cap B) \subset f(A) \cap f(B)$.

(b) $\Longrightarrow$ (d): If $A \cap B = \emptyset$ then by (b) and the initial remark we compute $f(A) \cap f(B) = f(A \cap B) = f(\emptyset) = \emptyset$.

(d) $\Longrightarrow$ (e): Put $C = B - A$, so that $A \cap C = \emptyset$. By (d), we first conclude that $f(A) \cap f(C) = \emptyset$ and thus that $f(C) = f(C) - f(A)$. But $C \subset B \implies f(C) \subset f(B) \implies f(C) - f(A) \subset f(B) - f(A)$, so $f(C) \subset f(B) - f(A)$. To see the other inclusion, first notice that $A \subset B \implies f(A) \subset f(B)$. It also implies that $B = A \cup C$. Thus $f(B) - f(A) =$

$f(A \cup C) - f(A) \subset f(C)$. Together these show that $f(B) - f(A) = f(C) = f(B - A)$.

(e) $\Longrightarrow$ (c): By solution 5, $A \subset f^{-1}[f(A)]$ for every $A \subset S$, so it is enough to show that $C = f^{-1}[f(A)] - A$ is the empty set for every A. But by (e) and solution 6, $f(C) = f\left(f^{-1}[f(A)] - A\right) = f\left(f^{-1}[f(A)]\right) - f(A) \subset f(A) - f(A) = \emptyset$, so that $C = \emptyset$ by the remark.

(c) $\Longrightarrow$ (a): If $y = f(x_1) = f(x_2)$, then $x_2 \in f^{-1}(\{y\}) = f^{-1}[f(\{x_1\})] = \{x_1\}$ by (c), so we must have $x_1 = x_2$. ■

Problem 9: Suppose that A is countable. Prove that if B is uncountable, then $B - A$ is uncountable.

Solution 9: We prove the contrapositive: if $B - A$ is countable, then B must be countable. But if $B - A$ is countable, then since A is countable the countable (in fact finite) union $(B - A) \cup A$ must also be countable. But then $B \subset (B - A) \cup A$ must be countable, since any subset of a countable set is countable. ■

Problem 10: Prove that every uncountable set contains a countably infinite subset.

Solution 10: If S is uncountable it is infinite, and thus nonempy, so choose $a_1 \in S$. Then $S_1 \stackrel{\text{def}}{=} S - \{a_1\}$ is also an infinite set, since otherwise $S \subset S_1 \cup \{a_1\}$ would be finite. We proceed to choose a_{n+1} from $S_n \stackrel{\text{def}}{=} S - \{a_1, \ldots, a_n\}$, which must also be nonempty for every n since S is infinite: if $S_n = \emptyset$ for some n then $S \subset \{a_1, \ldots, a_n\} \cup \emptyset$ must be finite. The sequence $\{a_n : n \in \mathbf{Z}^+\}$ is a countable infinite subset of S. ■